VECTOR MATH FOR 3D COMPUTER GRAPHICS

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Objectives: To learn and understand vector and matrix mathematics
from the point of view of computer graphics.

A. Points and Lines

Topics:
* Points
* Lines
* Triangles
* Vectors
* Representing points, lines, and vectors

A 3D graphics program builds a scene composed of three dimensional objects in three dimensional space. Then two dimensional images are produced from the 3D scene. The scene is represented as a data structure in computer memory. The 2D image is typically the picture on the face of a computer monitor.

The 3D objects are composed of points, lines, and polygons. You know what these are, but so that we can start from a common ground, this chapter reviews these concepts and introduces some of the vocabulary of computer graphics.

Two fundamental objects in geometry are points and lines. WHAT ARE THEY?

A **point** is a location in space. It has no other characteristics. It has no length, width, or thickness. It is pure location.

In geometry, the word point is not defined. It is one of the undefined primitives that are used to define other objects. Instead of defining "point" books give examples and hope that you somehow build up the concept from them.

At right is a picture that I hope will clarify this. The picture shows two spires atop a building. The picture is 2D, but think about the actual 3D building. Focus on the spike at the top of the closest spire.

What is the location of the very top of the spike? On the scale of the building, the end of the spike defines an exact location. We can conceptualize that location as a point.

A point is an idealization. On the scale of a building, the end of a spike is small enough to be considered an exact location, a point. But on the scale of your desktop, its end does not exactly define one location. For the scale of a desktop, the spike would have to taper to a pin point. Then, perhaps, you could consider its end to define a point.

But if you put that pin point under a microscope, then for that scale it is again too blunt. It would have to taper further, perhaps to the thickness of a bacterium, before the end could define a point. But now, on the scale of bacteria, the end of the spike is again too thick to exactly define a location.

So, points in 3D space are an idealization. For a computer model of a building, the sharp end of a spike is exact enough to be a point. The edges, planes, and other shapes that make up the building do not need to be specified to any greater accuracy.

In the physical world (the "real world"), the endpoints of the two spikes are at definite locations. Later on in these notes, locations will be specified using a coordinate frame. But for now let's just think of points in space. In 3D graphics programs you put points in space, and lines in space, and other things in space. And then project them into a 2D picture.

This mimics what has happened when the picture was produced with a camera. The ends of the spikes in the physical world define two 3D points. The whole scene was projected by a lens onto a 2D sheet of film. The ends of the spikes in the 2D picture define two 2D points.

QUESTION: What is the distance between the two points at the ends of the spikes?

In a 2D picture it is often hard to judge depth. When a 2D picture is produced from a 3D scene (either a real scene or a scene in a computer), a great deal of information is lost.

The real reason I asked the question was to trick you into thinking about the line segment between the two points. A line segment is the straight path between two points. It has no thickness. There is only one line segment between the two points.

Again, if you did not already have the concept of "line" the above definition would not be enough to learn it. The phrase "straight path" designates the concept that is being defined. In 3D graphics you mostly deal with line segments. In geometry, lines continue without end. A line segment is the part of a line between two points.

In geometry books the world line is usually not defined. It (like point) is an undefined concept used to define other objects. Books give pictures and examples of lines and hope that somehow you will come to understand what a line is by looking at them.

Often people say "line" when properly they should say "line segment." These notes will do that also, since line segments are so common.

At the scale of the building, a thin wire stretched between the two spikes is a close approximation to a line segment.

Again, this is a matter of scale. The wire is likely to be thicker than a line drawn by a pencil. On your desk the wire is too thick to be considered a line segment. Perhaps a thin, straight pencil line is thin enough to be considered a geometrical line. If that is too thick, then a length of spider silk stretched between two points might work.

But on an architectural level, taught wires can be considered line segments. You could build a geometrical model of the building using line segments to represent edges.

QUESTION: Mentally draw a triangle that connects the two spikes and the top left corner of the white railing.

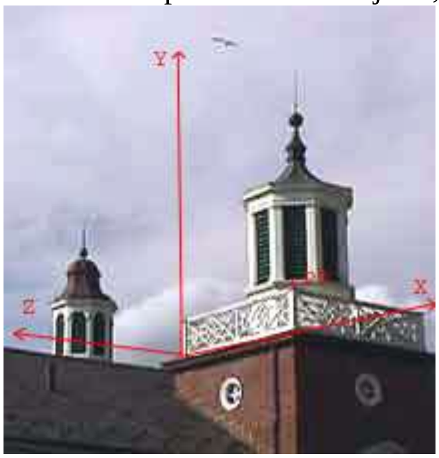
A triangle is made by connecting three points in 3D space with line segments. Unless the three points all lie on the same line, three points in space define a unique triangle. Triangles in space are very important in 3D graphics. The 3D objects that make up the scene are constructed out of triangles and other flat polygons.

The triangle in the 2D picture is a projection of the 3D triangle in the scene. The projection distorts the 3D triangle. The 3D triangle is much larger than the 2D triangle, and the angles are different.

Say that a 3D object in the scene is constructed of several triangles. Now project each triangle into a 2D picture. The result is the projected version of the 3D object. The white lattice work on the building is made up of many triangles. The image of the lattice in the picture shows all these triangles projected (by the lens and camera).

QUESTION: Do you need a coordinate frame (a system of labeling points with x, y, and z coordinates) to talk about points, lines, and plane figures?

You can think about the points, lines, and plane figures that compose the 3D scene without using a coordinate system. In fact, that is what we have been doing. But in computer graphics we need to represent these objects, somehow, and to manipulate them. To do this, we need a method to represent such objects.



Pick a convenient point in the 3D scene and call it the origin. Now define three lines called X, Y, and Z that go through the origin. Usually the three lines are perpendicular to each other. Each line is called an axis.

Pick a positive direction for each line. There are several choices here, but let's make the choice shown in the picture.

Now each point of the X axis is a unique distance (positive, negative, or zero) from the origin. For example, the corner of the building that lies on the X axis is at a distance of 8.0 feet (or so) from the origin.

QUESTION: The origin is at a height of zero. About how high is the end point of the closest spike?
ANSWER: I would say 12 feet.

Or you could say that for the endpoint of the spike, ***y=12.0***.

Any point in our 3D scene can be assigned a representation by measuring its distance along the three axes. For these notes (and for many computer graphics books), these three distances are put into a column matrix as follows.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

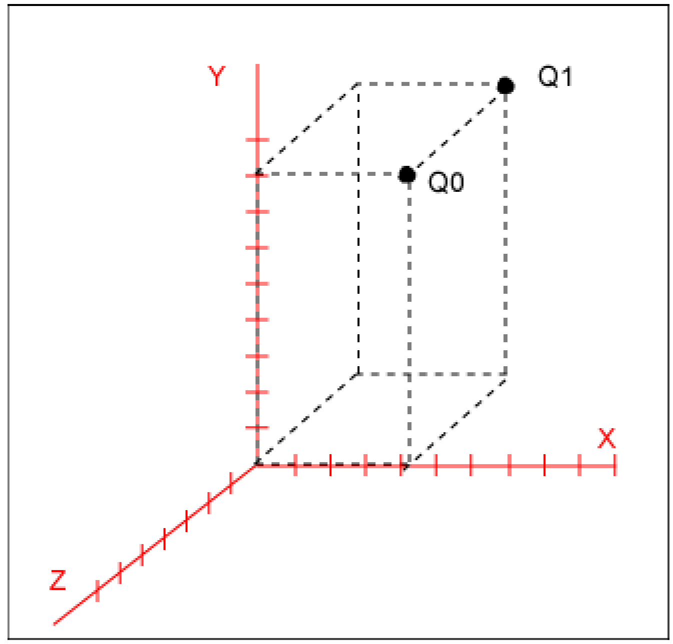
You are probably accustomed to putting the three coordinates into a triple such as (x, y, z). But it will turn out that for computer graphics it is more convenient to put them into a column, as above.

For example, the coordinates of the corner of the building that lies along the X axis are (about):

$$\begin{pmatrix} 8.0 \\ 0.0 \\ 0.0 \end{pmatrix}$$

QUESTION: What are the coordinates of the corner point of the white lattice along the top edge (labeled P0)?
ANSWER: Roughly,

$$\begin{pmatrix} 8.0 \\ 2.0 \\ 0.0 \end{pmatrix}$$



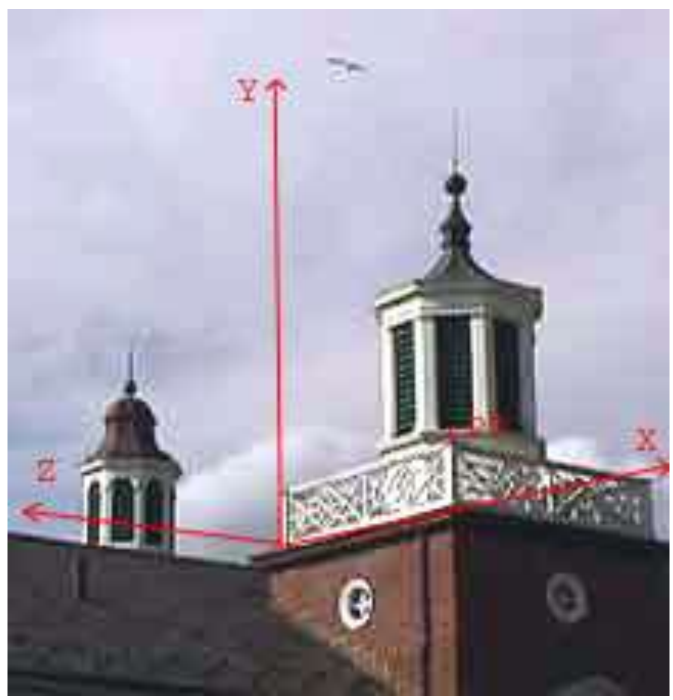
In determining coordinates, measure distance along lines that are parallel to the coordinate axes. The lines you use will form a rectangle or the sides of a box.

For example, in the figure, the black dotted lines used for point Q0 form a rectangle. Measuring the sides of the rectangle leads to the coordinates:

$$\begin{pmatrix} 4.0 \\ 8.0 \\ 0.0 \end{pmatrix}$$

It is harder to figure out the coordinates of Q1. The picture is 2D, but the scene it shows is 3D, so the Z axis is distorted. The dotted lines in the 3D scene form a box. All the angles are 90 degrees.

QUESTION: Estimate the coordinates of Q1.
ANSWER:
$$\begin{pmatrix} 4.0 \\ 8.0 \\ -4.0 \end{pmatrix}$$



Now back to our real-world scene. The white lattice is (I guess) eight feet long on each side.

QUESTION: Make a reasonable guess about the coordinates of the end of the closest spike.

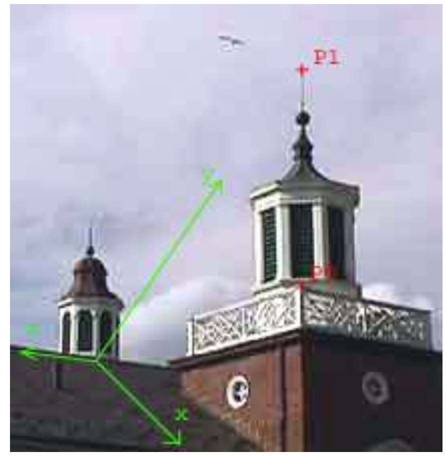
ANSWER:
$$\begin{pmatrix} 4.0 \\ 12.0 \\ -4.0 \end{pmatrix}$$

The **column matrix** is a mathematical object that represents the point using our chosen coordinate frame. The column matrix is not the point. The column matrix and the point are different things. One is used to represent the other, just as your name is used to represent you.

I hope this is obvious, but students who try to skip lightly through the mathematics of computer graphics sometimes are vague about this, to their eventual grief. Your computer graphics book will soon start representing the same point in several different ways. It is vital to keep separated the ideas of the geometrical point, and the several ways it might be represented.

Note: often people speak casually and say things like, "the point (4, 12, -4)," as if the triple of numbers were identical to the point. This is just for convenience. They really should say "the point that is represented in our chosen coordinate system by (4, 12, -4)."

QUESTION: Are the distances measured along the coordinate axes in feet or in meters?
ANSWER: It doesn't matter, as long as you are consistent



The coordinate frames used in math books are not usually placed in a real world scene, so the distances are not expressed in any particular unit. When you create your 3D world inside a computer you need to decide on the units you are using.

The picture shows our 3D world with a different coordinate frame placed in it. The points in the world are the same points as before. But now, with a different coordinate frame, they will have different representations.

In the previous frame, the point labeled P0 was given the representation:

$$\begin{pmatrix} 4.0 \\ 12.0 \\ -4.0 \end{pmatrix}$$

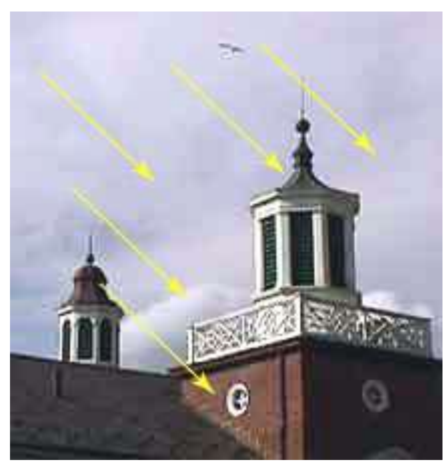
Now, the distances along the axes in the new (green) frame are different. The column matrix that represents P1 in the new frame is roughly:

(I am not doing math, here. I'm looking at the picture and guessing distances. Try to do the same. The goal is not to calculate anything but to think about points in space.)

The same point has a different representation in each frame. When you represent a point with a column matrix you have to know what frame is being used.

QUESTION: Estimate the coordinates of the point P0, on the top edge of the lattice.

ANSWER: $\begin{pmatrix} 3.0 \\ 8.0 \\ -4.0 \end{pmatrix}$ This is just an estimate.



Look at the picture of the building again. The scene is being illuminated by the sun, which is not in the picture. In the scene, sunlight is coming in diagonally from off scene to the left.

QUESTION: Mentally sketch some lines in the picture that show light from the sun.

As you look at the building, Light is flooding in from the upper left. The picture represents this sunlight as arrows. Although the picture is 2D, try to think of the 3D scene. The direction in which the light is traveling is shown by an arrow. The length of the arrow (let us say) represents the intensity of the light. Although there are several arrows in the picture, any one arrow is enough to represent the light and its intensity.

The direction of the light and its intensity can be represented as a vector. A **vector** is a geometrical object that has two properties: length and direction. Light from the sun has two properties: brightness and direction (we are disregarding color information.) The length of the vector proportional to the brightness of the light.

Wind, also, can be represented as a vector. Wind has direction and speed. A particle of dust travels in the direction of the wind. The length of the vector is proportional to the speed of the dust particle.

A vector does not have a position. The picture shows several arrows that represent the flood of light coming in from the sun. But the arrows could have been drawn anywhere as long as their direction and length was the same.

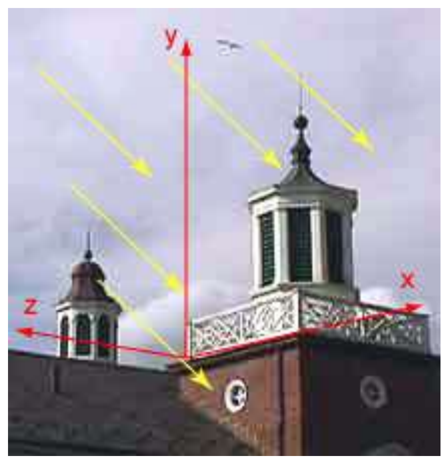
QUESTION: Do you suspect that a computer graphics program needs to represent light intensity and direction?
ANSWER: Yes.

Vectors are used for several purposes in 3D computer graphics. So they need to be represented in a way that programs can manipulate. And, just as you feared, this is done with numbers. The picture now includes our coordinate frame.

A vector is represented with a column matrix (as are points). It is somewhat confusing that both points and vectors are represented with the same thing. But this will prove to be convenient for computer manipulation.

To represent a vector as a column matrix:

1. Choose a coordinate frame.
2. Calculate the X length (positive or negative) from the tail of the vector to the tip.
3. Calculate the Y length (positive or negative) from the tail of the vector to the tip.
4. Calculate the Z length (positive or negative) from the tail of the vector to the tip.
5. Put those three numbers into a column.



A convenient way to do this is to position the tail of the vector at the origin, and then just read off the x, y, z values at the tip.

QUESTION: Try to do this with the picture.

1. For the arrows in the picture, how many feet in the X direction is the tip from the tail? ______
2. How many feet in the Y direction is the tip from the tail? ______
3. How many feet in the Z direction is the tip from the tail? ______

ANSWER: It is hard to do anything but guess. My guesses are:

1. For the arrows in the picture, how many feet in the X direction is the tip from the tail? 3 ft
2. How many feet in the Y direction is the tip from the tail? -4 ft
3. How many feet in the Z direction is the tip from the tail? -3 ft

So the vector that gives the intensity and direction of sunlight in the picture is:

$$\begin{pmatrix} 3.0 \\ -4.0 \\ -3.0 \end{pmatrix}$$

The column matrix represents the vector using our chosen coordinate frame. The column matrix is not the vector. If we had picked a different coordinate frame, the vector in the physical scene would be the same, but its representation using the new frame would change.

B. Vectors, Points and Column Matrices

This section discusses the objects of computer graphics—vectors and points—and how they are represented in a computer—as column matrices. A **column matrix** is a mathematical object that has many uses besides its use in computer graphics. These notes discuss only how they are used in computer graphics.

Topics:
* Computer graphics as modeling and viewing
* Geometrical points and vectors
* Column and row matrices
* Calculating displacements
* Equality of column matrices
* Names for column matrices
* Representing points with column matrices

Geometrical points and vectors are essential in computer graphics. They must be represented in a way that is easy to manipulate. A column matrix is the usual choice for this.

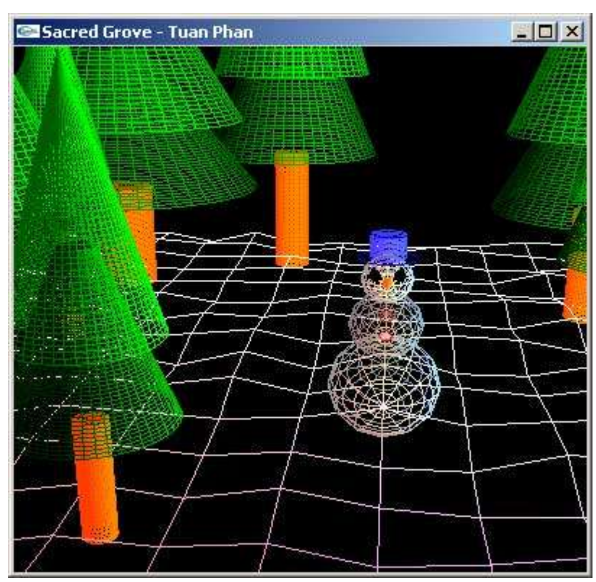
Some graphics books use the term column vector for the object that these notes call a column matrix. This is just a variation in terminology and does not affect the concepts or formulas presented here.

A worse problem is that some books use row matrices. Column matrices and row matrices are equivalent, but equations written using row matrices are not the same as those written using column matrices. The differences are easy to adjust for, but annoying.

QUESTION: (Review of previous topic:) What two types of geometrical objects are represented with column matrices?

ANSWER: (1) Points, and (2) Vectors.

Computer graphics consists of two activities: (1) Creating an imaginary world inside a computer, and (2) producing two dimensional images of that world from various viewpoints. A graphics program is like a tourist wandering through a fantastic landscape taking pictures with a camera. With computer animation, the virtual tourist is equipped with a movie camera.



The imaginary landscape is built of objects in three dimensional space. Each object is defined by the points and lines that lie on its surface. Here, for example, is a winter landscape (done by Tuan Phan).

The model consists of points and the line segments that connect them. To make a realistic picture, more work is needed to fill in the area between line segments and to apply lighting.

QUESTION: Can a sphere be modeled with points and lines?

ANSWER: Sure. Look at the body of the snowman. When the polygons are filled in the shape looks like a sphere.

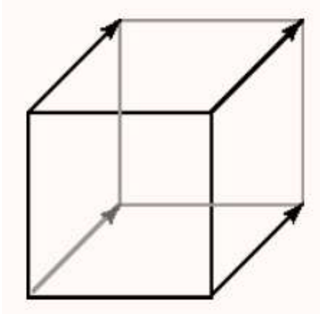
With 3D computer graphics packages (such as OpenGL), you can ask for models of spheres, cylinders, and cones and place them in your scene where you want. OpenGL will also automatically fill in the polygons and apply a lighting model to produce realistic shading.

QUESTION: Could our virtual tourist wander into the virtual snowscape and take a different picture?
ANSWER: Yes.

The model of the snowscape puts various objects in a 3D space. Our virtual tourist could wander over to the right of the snowman and make another 2D picture with the virtual camera. The picture below was produced with the same model and lighting, but with a different camera placement.

In geometry, a point is a location in space. A point does not have any size, its only property is location. In computer graphics, a point is usually the vertex of a 3D figure.

A geometrical vector has two properties: length and direction. A vector does not have a fixed location in space. If it did, it would be a line segment. It seems odd to talk about something that does not have a location but this makes 3D computer graphics easier.



This combination of "distance and direction" is sometimes called a **displacement**. Often the same displacement (i.e. just one displacement) is applied to each of several points. For example, consider a cube. The front face contains four vertices (four points). If you move the same distance and direction from each of these points you reach the four vertices of the back face.

The "same distance and direction" is a vector, shown here as a line with an arrow head. The diagram shows this one vector four times, once for each point of the face.

QUESTION: Say that it is Spring Break and you are at the beach and the Sun is shining down. Is the light from the Sun shining down in the same direction for everyone on the beach?

ANSWER: Yes. The "direction to the Sun" is the same for everyone on the beach. It is a vector.

Sometimes (as in the question) we are interested only in direction, not location, nor length. Use a vector for this. Its length does not matter. Often it will be given a length of one.

A geometric vector may be represented with a list of numbers called a **column matrix**. A column matrix is an ordered list of numbers written in a column. Here is an example of a column matrix:

$$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$$

Each number of the column matrix is called an element. The numbers are real numbers. The number of elements in a vector is called its **dimension**. A **row matrix** is an ordered list of numbers written in a row. Here is an example of a row matrix:

(12.5, -9.34)

To be consistent, our vectors will always be represented with column matrices. Some books represent vectors with row matrices, which makes no fundamental difference, but slightly changes some mathematical formulas.

QUESTION: How many elements are in each column matrix?

Column Matrix	$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -13.45 \\ 92.4 \\ 2.0 \\ -42.01 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$
Dimension	. ? .	. ? .	. ? .

The elements of a column matrix can be *variables*:

$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$	$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

The first element in a column matrix is sometimes given the index "0", and sometimes "1".

QUESTION: Is the column matrix $\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$ the same as the column matrix $\begin{bmatrix} 2.9 \\ 0.0 \\ -4.6 \end{bmatrix}$?

ANSWER: No. A column matrix is an ordered list of numbers. This means that each position of the column matrix contains a particular number (or variable.)

Column matrices are awkward in printed text. It is common to print a column matrix like this:

$$(2.9, -4.6, 0.0)^T$$

The "T" stands for transpose, which means to change rows into columns.

QUESTION: Is (1.2, -3.9, 0.0) equal to $(1.2, -3.9, 0.0)^T$?
ANSWER: No. The first matrix is a row matrix, and the second matrix is a column matrix (printed in a row, but the "T" means it is really a column matrix). Row matrices and column matrices are different types of objects, and cannot be equal.

Here is what it takes for two column matrices to be equal:

1. Both matrices must be column matrices.
2. Both must have the same dimension (number of elements).
3. Corresponding elements of the matrices must be equal

Here is what it takes for two row matrices to be equal:

1. Both matrices must be row matrices.
2. Both must have the same dimension (number of elements).
3. Corresponding elements of the matrices must be equal

QUESTION: Are the following row matrices equal?

$$(8.0, -1.63, 7.0, 0.0)$$
$$(8.0, -1.63, 7.0, 1.0)$$

ANSWER: No.

Only matrices of the same type can be compared. You can compare two three-dimensional column matrices, or two four-dimensional row matrices, and so on. It makes no sense to compare a three-dimensional row matrix to a three-dimensional column matrix. For example:

$$(6, 8, 12, -3)^T = (6, 8, 12, -3)^T$$
$$(6, 8, 12, -3) = (6, 8, 12, -3)$$
$$(6, 8, 12, -3) \neq (-2.3, 8, 12, -3)$$
$$(6, 8, 12, -3)^T \neq (6, 8, 12, -3)$$
$$(6, 8, 12, -3)^T \neq (6, 8, 12)^T$$

QUESTION: Are the following column matrices equal?

$$(1.53, -0.03, 9.03, 0.0, +8.64)^T$$
$$(1.53, -0.03, 9.03, 1.0, -8.64)^T$$

ANSWER: No.

It is useful to have names for matrices. Usually a bold lower case letter is used for a column or a row matrix:

$$a = (1.2, -3.6)$$
$$x = (x1, x2, x3, x4)$$
$$r = (r0, r1)^T$$

It is conventional to use names from the start of the alphabet for column matrices whose elements we know (like a above), and to use names from the end of the alphabet for column matrices whose elements are variables.

Often the names of column matrix elements are subscripted versions of the name of the whole column matrix (like column matrix r and its elements r0 and r1).

Bold face is hard to do with pencil or chalk, so instead an arrow or a bar is placed over the name:

$$\bar{x} \quad \vec{x}$$

QUESTION: Say that you know:

$\mathbf{x} = (x_1, x_2)$
$\mathbf{y} = (3.2, -8.6)$
$\mathbf{x} = \mathbf{y}$

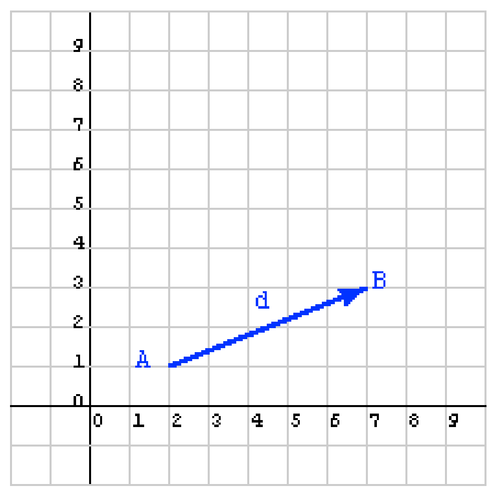
What must be true about x_1 and x_2?

ANSWER: $x_1 = 3.2$, and $x_2 = -8.6$

Column matrices are used to represent vectors and also used to represent points. In two dimensions, the same data type (two dimensional column matrices) is used to represent two different types of geometrical objects (points and vectors). This is awkward. Later on, this situation will be corrected.

The figure shows a displacement vector representing the difference between two points in the x-y plane. (For now, the examples are in two dimensional space; three dimensional space will come later).

- point A: x=2, y=1. As a column matrix: $(2, 1)^T$
- point B: x=7, y=3. As a column matrix: $(7, 3)^T$



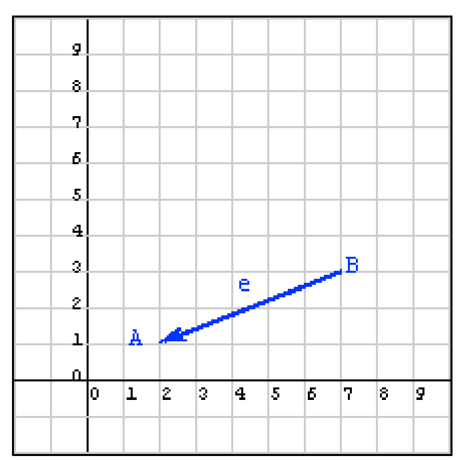
Computing the displacement from A to B can be done as two separate problems:

- The x displacement is the difference in the X values: 7-2 = 5
- The y displacement is the difference in the Y values: 3-1 = 2

The displacement vector expressed as a matrix is:

$\mathbf{d} = (5, 2)^T$

Displacement vectors are often visualized as an arrow connecting two points. In the diagram point A is the tail of the vector and point B is the tip of the vector. (However, recall that vectors do not have a position, so this is just a convenient place to draw it, not where it is.)



QUESTION: The column matrix d represents the displacement vector from point A to point B. What column matrix represents displacement from point B to point A?

ANSWER: The displacement from point B to point A is:

* The x displacement: 2-7 = -5
* The y displacement: 1-3 = -2
So the column matrix representing the displacement is: $e = (-5, -2)^T$

When the points are visited in the opposite order, the displacement vector points in the opposite direction. In the column matrix each element is -1 times the old value.

The displacement from A to B is different from the displacement from B to A. Think of displacement as "directions on how to walk from one point to another." So, if you are standing on point A and wish to get to point B, the displacement $(5, 2)^T$ says "walk 5 units in the positive X direction, then walk 2 unit in the positive Y direction."

Of course, to get from point B to point A you need different directions: the displacement $(-5, -2)^T$ says "walk 5 units in the *negative* X direction, then walk 2 unit in the *negative* Y direction," which puts you back on point A.

The displacement from point **Start** to point **Finish** :

$$\textbf{displacement} = \textbf{(Finish x - Start x , Finish y - Start y)}^T$$

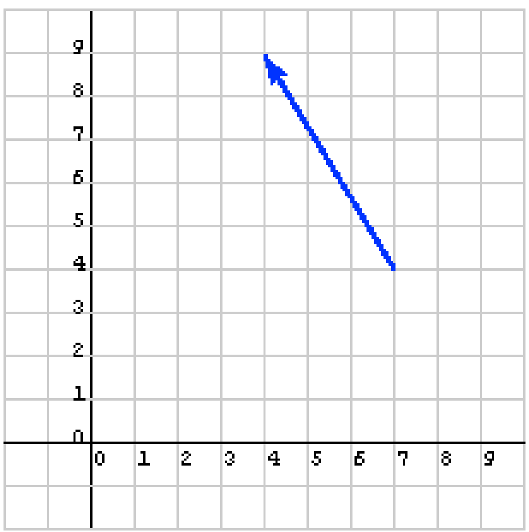
QUESTION: Say that point C is x=4, y=2 and that point D is x=3, y= 5 . What column matrix represents the displacement from C to D?

ANSWER:

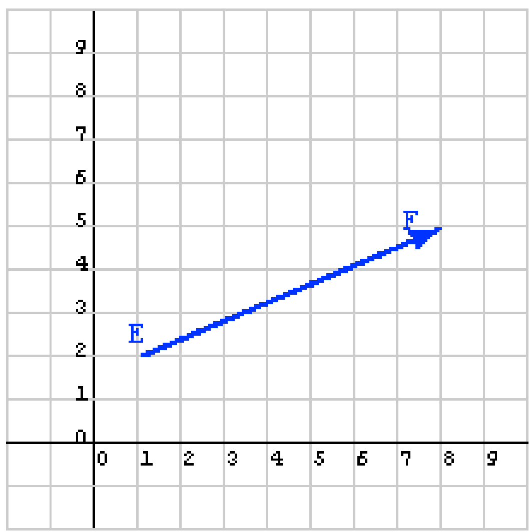
* Finish X - Start X = 3 - 4 = -1
* Finish Y - Start Y = 5 - 2 = 3

So the column matrix is $(-1, 3)^T$

The first diagram shows a vector between two points. The numerical values can be read off of the graph by counting the number of squares from the tail of the vector to the tip: $(-3, 5)^T$.



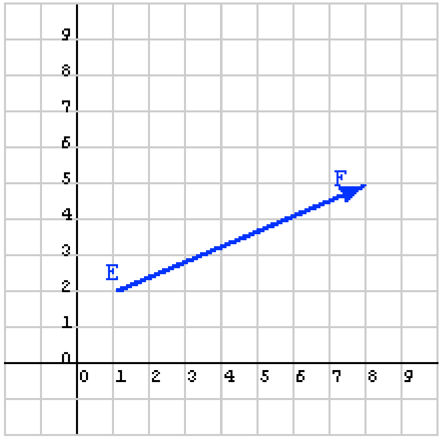
First Diagram



Second Diagram

QUESTION: Do that with the second diagram. What is the displacement from E to F?

To check your answer, start at the beginning point and follow the directions: walk 7 in X, then walk 3 in Y. If your answer is correct, you end up at the ending point.



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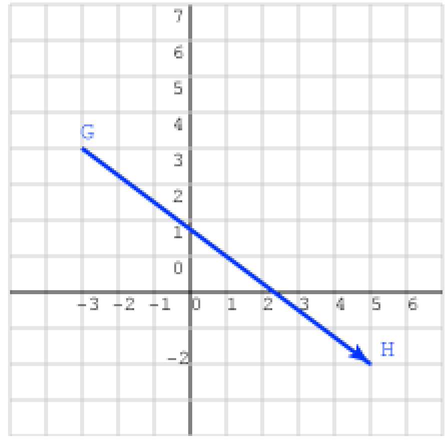

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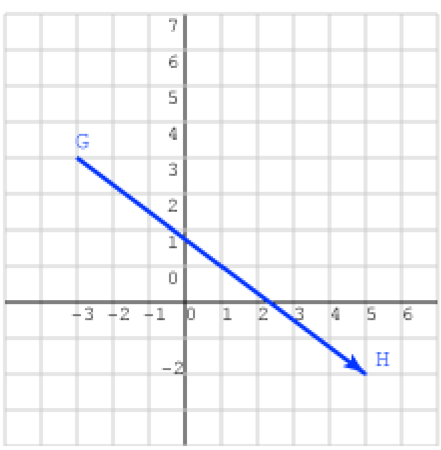
QUESTION: What is the displacement from point G, $(-3, 4)^T$ to point H, $(5, -2)^T$?

ANSWER: Subtracting values of the start G from corresponding values of the finish H gives: $(8, -6)^T$

Now calculate the displacement from M to N in the diagram on the right. Subtracting values of the start N from corresponding values of the finish M gives: $(8, -6)^T$. This is the same as for the first diagram.

Geometrically, the displacement vector from G to H is the same as the displacement vector from M to N. Using the rule for column matrix equality, the two column matrices are equal. This makes sense because in walking from point G to H you go the same distance and direction as in walking from M to N. The diagrams show the displacements with the same length and direction (but different starting points):

Vectors have no location. In a diagram it is common to draw a vector as an arrow with its tail on one point and its tip on another point. But any arrow with the same length and direction represents the vector.



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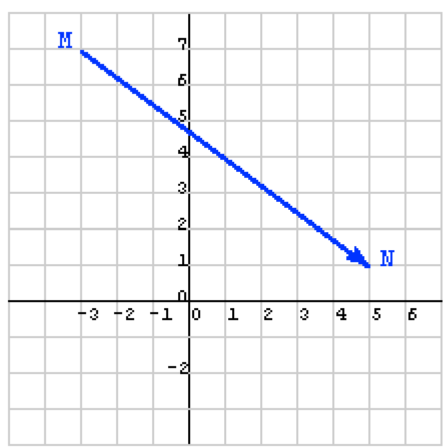

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QUESTION: Is the displacement between two points unique?

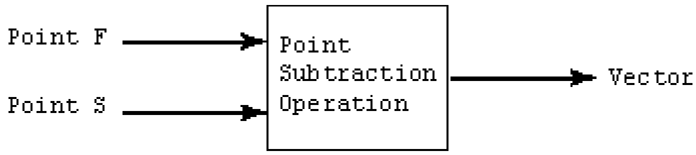
ANSWER: Yes.

The previous formula for calculating a displacement vector can be written as:

Displacement from point S (start) to point F (finish) =

$F - S = (X_f , Y_f)^T - (X_s , Y_s)^T = (X_f-X_s , Y_f-Y_s)^T$

In this operation two points are used to produce one vector:



QUESTION: (Thought question:) If one three dimensional point is subtracted from another, is the result a vector?

ANSWER**:** Yes. These ideas work in 3D as well as 2D.

Practice with Displacements

It would be a pity to be talking about math without a story problem: Amy the ant at point $(8, 4)^T$ is lost. Find a displacement that will bring her to her friend Emily at $(2, 2)^T$.

QUESTION: Through what displacement should Amy move to find her friend Emily?

ANSWER: By subtracting points:

> **Emily - Amy = $(2, 2)^T - (8, 4)^T = (-6, -2)^T$.** (Of course by counting squares of the graph paper you will get the same answer.)

With the story problem, there is a temptation to avoid negative numbers and to incorrectly compute the displacement. But it is perfectly OK to have negative displacements; the "negative" part is needed to show the direction.

Another thing to keep in mind is that the elements of a column matrix are real numbers. Examples often use integers, but that is just to make things easy. There is nothing wrong with the column matrix $(-1.2304,9.3382)^T$.

QUESTION:

- point B = (4.75, 6.23)
- point A = (1.25, 4.03)

What is the displacement from point A to point B?

ANSWER**: The displacement is $(4.75, 6.23) - (1.25, 4.03) = (3.50, 2.20)^T$**

C. Column and Row Matrix Addition

Here are some of the terms we have been using:

- **point** — a geometric object; a location in 3D (or 2D) space.
- **vector** — a geometric object that has properties of direction and length, but not location.
- **column matrix** — an ordered list of numbers arranged into a column.
- **row matrix** — an ordered list of numbers arranged into a row.
- **element** — one of the numbers that makes up a column or row matrix.
- **dimension** — the number of elements in a column or row matrix.
- **displacement** — the difference between two locations, expressed as a vector.

This section discusses how column and row matrices are added or subtracted.

QUESTION: (Review question) What is it called when a column matrix is "flipped" on its side?

ANSWER: **Transposition.**

Even if a row matrix and a column matrix are the same dimension and contain the same elements, they are considered different types. A "T" superscript is used when a column matrix is written as a row of numbers. The first element of a column matrix is the topmost (corresponding to the leftmost element when written as a row.)

The picture shows two different ways of writing out the same column matrix. When the column matrix is written in a row the superscript "T" shows that it is really a column matrix. The elements of the column matrix have the same subscripts no matter how the column matrix is displayed.

$$\begin{pmatrix} -3.5 \\ 97.2 \\ -24.9 \end{pmatrix} \qquad (-3.5, 97.2, -24.9)^T$$

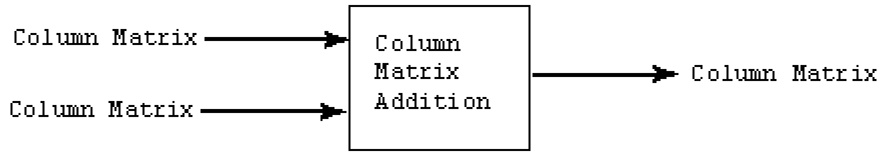
A T superscript on column matrix means to flip the column into a row, resulting in a row matrix. (Some books use a lower-case t.)

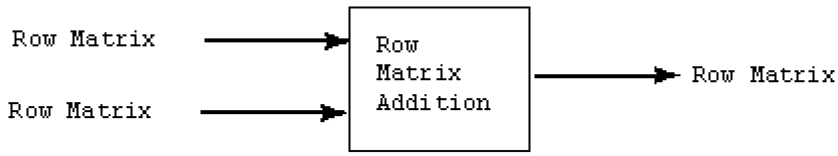
QUESTION 2: What is: $(1, 2, 3)^{T\,T}$?

ANSWER: $(1, 2, 3)^{T\,T} = (1, 2, 3)$

Transposing twice gives you what you started with. This seems a bit dumb right now, but later on when you are doing algebraic manipulation it might help to remember this.

A column matrix added to another column matrix of the same dimension yields another column matrix (with the same dimension). A similar statement is true for row matrices.





Addition is done by adding corresponding elements of the input matrices to produce each corresponding element of the output matrix.

$(1, 2, 3) + (10, 20, 30) = (11, 22, 33)$

$(42, -12)^T + (8, 24)^T = (50, 12)^T$

$(9.2, -8.6, 3.21, 48.7) + (-2.1, 4.3, 1.0, 2.3) = (7.1, -4.3, 4.21, 51.0)$

$(32.98, -24.71, 9.392)^T + (-32.98, +24.71, -9.392)^T = (0, 0, 0)^T$

If a and b are matrices of the same type, then a + b = c means that each element ci = ai + bi

QUESTION: Do the following problem: $(2, -2)^T + (8, 6)^T =$ _____.

ANSWER: $(2, -2)^T + (8, 6)^T = (10, 4)^T$

The matrices you add must be of the same dimension and same row or column type. When you add matrices, each dimension is handled independently of the others. For example, in adding two 3D column matrices, you have three separate additions using ordinary arithmetic. The first elements of the operand matrices are added together to make the first element of the result matrix.

QUESTION: Do the following two problems:

(8, 4, 6)T + $(2, -2, 9)^T$ =

(2, -2, 9)T + $(8, 4, 6)^T$ =

ANSWER:

$(8, 4, 6)^T + (2, -2, 9)^T = (10, 2, 15)^T$

$(2, -2, 9)^T + (8, 4, 6)^T = (10, 2, 15)^T$

Matrix addition is commutative. This means that a + b = b + a.

This works for both row and column matrices of all dimensions. It is also true that:

a + b + c = b + c + a = c + a + b = ...

In other words, the order in which you add matrices does not matter. At least for addition, matrices work the same way as numbers, since, of course, 1 + 2 = 2 + 1.

Matrices and numbers usually do NOT work the same way! But they do for addition.

QUESTION: Contemplate this problem: $(-1, -2, 3)^T + (1, 2, -3)^T =$

ANSWER: $(-1, -2, 3)^T + (1, 2, -3)^T = (0, 0, 0)^T$

A matrix with all zero elements is sometimes called a **zero matrix**. The sum of a zero matrix and a matrix **a** of the same type is just **a**.

In symbols the zero matrix is written as **0** (bold face zero) which is different than 0, the real number zero. The context in which you see **0** determines whether it is a row or a column and how many elements it has. So if you see

$$(73.6, -41.4)^T + \mathbf{0}$$

you can assume that 0 is the right type of matrix for the expression.

QUESTION: Do the following problem:

(2, 5, -9) + (-32.034, 94.79, 201.062) + (-2, -5, 9) =

ANSWER:

(2, 5, -9) + (-32.034, 94.79, 201.062) + (-2, -5, 9)

= (-32.034, 94.79, 201.062) + $\underline{(-2, -5, 9) + (2, 5, -9)}$

= (-32.034, 94.79, 201.062) + $\underline{(0, 0, 0)}$

= (-32.034, 94.79, 201.062)

Matrix addition is associative. This means that **(a + b) + c = a + (b + c)**.

This says "first add **a** to **b** then add that result to **c**." The result will be the same as if you did "add **a** to the result of adding **b** with **c**." This works for both row and column matrices of all dimensions.

QUESTION: Solve the problem: (25.1, -19.6) + (-5.0, 9.0) + (12.4, 8.92) + (-20.1, 10.6) =

ANSWER:

```
( 25.1, -19.6 ) + ( -5.0, 9.0 ) + ( 12.4, 8.92 ) + ( -20.1, 10.6 )  =
                  -------------                    ---------------
                                \                 /
                                  \             /
 ( 25.1, -19.6 ) + ( 12.4, 8.92 ) + ( -25.1, 19.6 )
---------------                     ---------------
              \                     /
               \                  /
                \               /
                   ( 0.0, 0.0 )   +  ( 12.4, 8.92 ) = ( 12.4, 8.92 )
```

In computer science terms, the "+" symbol is *overloaded,* which means that the operation called for depends on the type of operands. For example:

> $1.34 + -9.06$
> + means addition of real numbers
>
> $(84.02, 90.31)^T + (-14.23, 10.85)^T$
> + means addition of 2D column matrices

The following have no meaning:

> $34.5 + (84.02, 90.31)^T$
> can't add a number to a matrix
>
> $(84.02, 90.31) + (-14.23, 10.85)^T$
> can't add a row matrix to a column matrix
>
> $(84.02, 90.31) + (-14.23, 10.85, 32.75)$
> can't add matrices of different dimensions

These problems are clear when the elements are written out, as above, but it is less clear when variable symbols are used:

> $a + \mathbf{x}$ <-- can't add a number to a matrix
>
> $\mathbf{x} + \mathbf{y}$ <-- make sure that both are of the same type

There are some strange-looking things you can do, such as in the following:

QUESTION: Perform the following addition:

$$(4.5, x_1, w) + (-2.3, 3, y_2) =$$

ANSWER:

$$(4.5, x_1, w) + (-2.3, 3, y_2) = (2.2, x_1+3, w+y_2)$$

Two matrices of the same type can be subtracted to produce a third matrix of the same type. As you probably imagine, subtracting two matrices means subtracting the corresponding elements, *being careful to keep the elements in the same order:*

$$(10, 8, 12) - (2, 14, 9) = (10 - 2, 8 - 14, 12 - 9) = (8, -6, 3)$$

If **a** and **b** are matrices of the same type, $\mathbf{a} - \mathbf{b} = \mathbf{c}$ means that each element $c_i = a_i - b_i$

QUESTION: Do these two problems:

> $(22, 5, -12) - (10, -5, 3) =$
>
> $(10, -5, 3) - (22, 5, -12) =$

ANSWER:

(22, 5, -12) - (10, -5, 3) = (22 - 10, 5 - (-5), -12 - 3) = (12, 10, -15)

(10, -5, 3) - (22, 5, -12) = (10 - 22, (-5) - 5, 3 - (-12)) = (-12, -10, 15)

Matrix subtraction is *NOT commutative*.

This means that you can't change the order when doing **a - b**.

The **negative** of a matrix is this:

-a means negate each element of **a**.

So, for example:

-(22, 5, -12) = (-22, -5, 12)

$-(19.2, 28.6, 0.0)^T = (-19.2, -28.6, 0.0)^T$

In symbols: if **a** is (a_0, a_1, ..., a_2) then **-a** means ($-a_0$, $-a_1$, ..., $-a_2$).

QUESTION 10:

Perform the following subtraction:

$(-7.2, -98.6, 0.0)^T - (-2.2, -2.4, 3.0)^T$

ANSWER:

$(-7.2, -98.6, 0.0)^T - (-2.2, -2.4, 3.0)^T =$
$= (-7.2 -(-2.2), -98.6 -(-2.4), 0.0 - 3.0)^T$
$= (-5.0, -96.2, - 3.0)^T$

You may find the following an easier way to do the problem:

$(-7.2, -98.6, 0.0)^T - (-2.2, -2.4, 3.0)^T =$
$(-7.2, -98.6, 0.0)^T + (+2.2, +2.4, -3.0)^T =$
$(-5.0, -96.2, - 3.0)^T$

The "outside -" was used to negate the second matrix, then the resulting two matrices were added. In symbols:

a - b = a + (-b)

This can be more useful to remember than it at first appears. You can negate matrices so the problem is one of matrix addition, then rearrange the addition (because addition is commutative):

a - b + c - d = a + (-b) + c + (-d)
= (-d) + a + c + (-b)
= any rearrangement you want

Some other facts are:

a + (-a) = 0
a - a = 0

Notice that the **0** means the *zero matrix*, the matrix of the same type as **a**, but with all elements zero.

QUESTION:

Do the following operation:

(4, -5, 6.2) + (-43.132, 13.6, 86.5) - (4, -5, -4.8) =

ANSWER:

This looks like another trap. Rather than blindly rushing in and calculate, try rearranging things:

(4, -5, 6.2) + (-43.132, 13.6, 86.5) - (4, -5, -4.8) =

(4, -5, 6.2) + (-43.132, 13.6, 86.5) + (-4, 5, 4.8) =

(4, -5, 6.2) + (-4, 5, 4.8) + (-43.132, 13.6, 86.5) =

(0, 0, 11) + (-43.132, 13.6, 86.5) = (-43.132, 13.6, 97.5)

You would probably skip a few steps if you were doing this mentally.

QUESTION:

What is the sum of the following three displacements:

d = $(-12.4,\ 14.8,\ 0.0)^T$
e = $(6.2, -10.2,\ 17.0)^T$
f = $(6.2,\ -4.6, -17.0)^T$

ANSWER:

d = $(-12.4,\ 14.8,\ 0.0)^T$
e = $(6.2, -10.2,\ 17.0)^T$
f = $(6.2,\ -4.6, -17.0)^T$

$(0.0,\ 0.0\ \ 0.0)^T$

QUESTION:

Find x, y, and z so that the following is true:

a = (8.6, 7.4, 3.9)
b = (4.2, 2.2, -3.0)
c = (x, y, z)

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

ANSWER:

$\mathbf{a} + \mathbf{b} + \mathbf{c} = (12.8, 9.6, 0.9) + (x, y, z) = (12.8+x, 9.6+y, 0.9+z) = (0, 0, 0)$

So it must be that:

$12.8 + x = 0;\ \ x = -12.8$
$9.6 + y = 0;\ \ y = -9.6$
$0.9 + z = 0;\ \ z = -0.9$

The problem was: find the elements of **c** when

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

If you didn't know they were matrices, you might have been tempted to work this using real number algebra:

$$\begin{aligned}
\mathbf{a} + \mathbf{b} + \mathbf{c} &= \mathbf{0} \\
\mathbf{a} + \mathbf{b} &= \mathbf{-c} + \mathbf{0} \\
(\mathbf{a} + \mathbf{b}) &= \mathbf{-c} \\
-(\mathbf{a} + \mathbf{b}) &= \mathbf{c}
\end{aligned}$$

In fact, this works. As long as every matrix is of the same type, and the operations are only "+" or "-", you can pretend that you are doing ordinary algebra. Notice that the last equations means "add **a** with **b**, then negate the result to get **c**."

To see this, look at just the first elements of the matrices:

$$\begin{aligned}
(a_0, ...) + (b_0, ...) + (c_0, ...) &= (0, ...) \\
(a_0, ...) + (b_0, ...) &= -(c_0, ...) + (0, ...) \\
(a_0, ...) + (b_0, ...) &= -(c_0, ...) \\
(a_0 + b_0, ...) &= -(c_0, ...) \\
-(a_0 + b_0, ...) &= (c_0, ...) \\
-(a_0 + b_0) &= c_0
\end{aligned}$$

If this is too ugly for you this early in the morning, mentally erase some of the junk:

$$\begin{aligned}
a_0 + b_0 + c_0 &= 0 \\
a_0 + b_0 &= -c_0 + 0 \\
a_0 + b_0 &= -c_0 \\
a_0 + b_0 &= -c_0 \\
-(a_0 + b_0) &= c_0
\end{aligned}$$

Of course the other elements follow the same pattern so the result is true for the matrix as a whole.

QUESTION:

Find c_0 and c_1 so that the following is true:

$\mathbf{a} = (-4, 2)^T$
$\mathbf{b} = (8, 3)^T$
$\mathbf{c} = (c_0, c_1)^T$

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

ANSWER:

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$
$\mathbf{a} + \mathbf{b} = \mathbf{-c}$

$(4, 5)^T = -(c_0, c_1)^T$
$-(4, 5)^T = (c_0, c_1)^T$
$(-4, -5)^T = (c_0, c_1)^T$

$c_0 = -4$
$c_1 = -5$