College of Computer Science
DMMMSU-SLUC
Agoo, La Union

CHAPTER 2
Basic Structures

PROF. KATHRINA T. DACANAY
Instructor, Discrete Structures

COURSE OUTLINE

- Introduction
- Sets
  - Set Operations
- Functions
- Sequences
- Summations
- Matrices

INTRODUCTION

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, sequences, summations and matrices.

SETS

INTRODUCTION

Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. Set is the fundamental discrete structure on which all other discrete structures are built. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties.

SET DEFINED

A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

TWO WAYS TO DESCRIBE A SET

1. **Roster Method.** It lists all the members of a set, when possible. We use a notation where all members of the set are listed between braces. Example: are V = {a, e, i, o, u} and O = {1, 3, 5, 7, 9}. Sometimes, the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then ellipses (. . .) are used when the general pattern of the elements is obvious. Example: A = {a, b, c, d, e, … , z} and E = {2, 4, 6, 8, … , 100}.

TWO WAYS TO DESCRIBE A SET

2. **Set Builder.** It characterizes all those elements in the set by stating the property or properties they must have to be members. Example: V = {x | x is a vowel} and O = {x | x is an odd positive integer less than 10}.

EQUAL SETS

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets. The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

EMPTY SET

There is a special set that has no elements called the empty set or null set, and is denoted by ∅ or { }. Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

SINGLETON SET

A set with one element is called a singleton set. A common error is to confuse the empty set Ø with the set {Ø}, which is a singleton set. The single element of the set {Ø} is the empty set itself.

VENN DIAGRAM

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams, the universal set U, which contains all the objects under consideration, is represented by a rectangle. Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent elements of the set. Venn diagrams are often used to indicate the relationships between sets.

VENN DIAGRAM

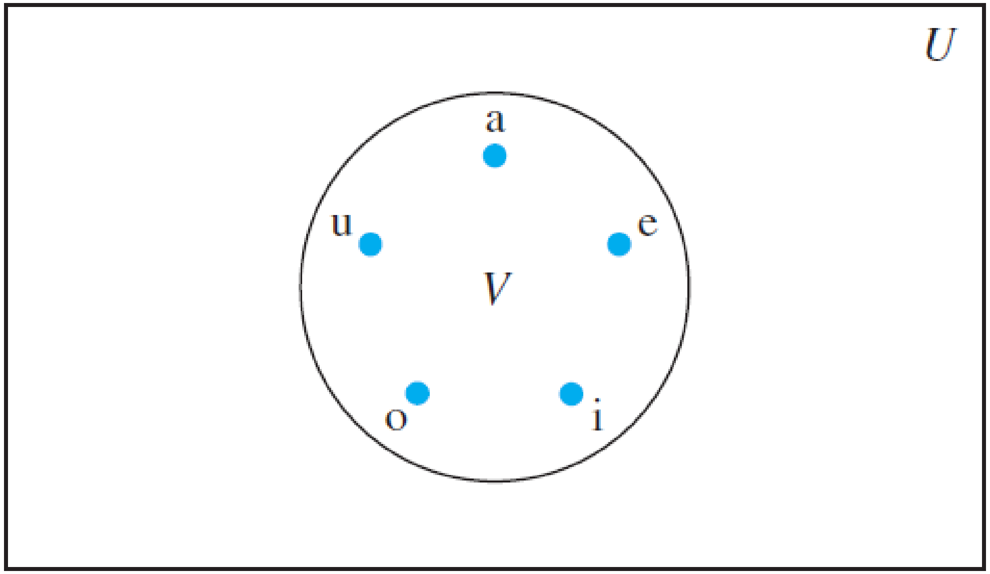


FIGURE 1 Venn Diagram for the Set of Vowels

SUBSETS

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation A ⊆ B to indicate that A is a subset of the set B.

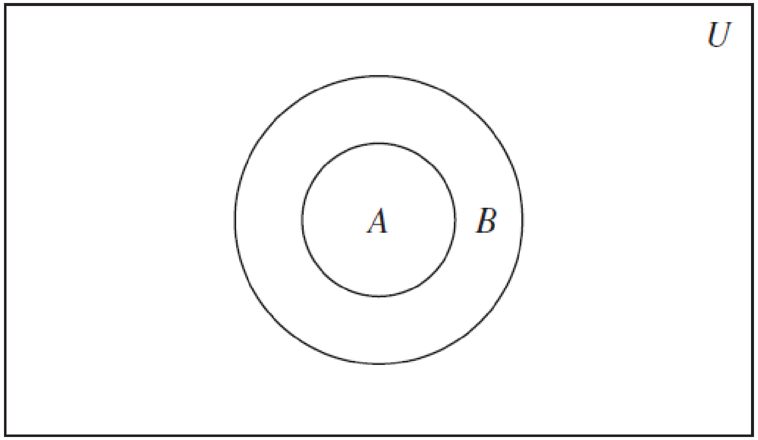


FIGURE 2 Venn Diagram Showing that *A* is a Subset of *B*.

SIZE OF A SET

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by |S|. Example: Let A be the set of odd positive integers less than 10, then |A| = 5 and let S be the set of letters in the English alphabet, then |S| = 26. A set is said to be infinite if it is not finite.

POWER SETS

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S). Example, the power set of {0,1,2} is the set of all subsets of {0,1,2}. Hence, P({0,1,2}) = {ø, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}. Note that the empty set and the set itself are members of this set of subsets. The empty set has exactly one subset, namely, itself. Consequently, P(ø) = {ø}. The set {ø} has exactly two subsets, namely, ø and the set {ø} itself. Therefore, P({ø}) = {ø, {ø}}. If a set has n elements, then its power set has 2^n elements.

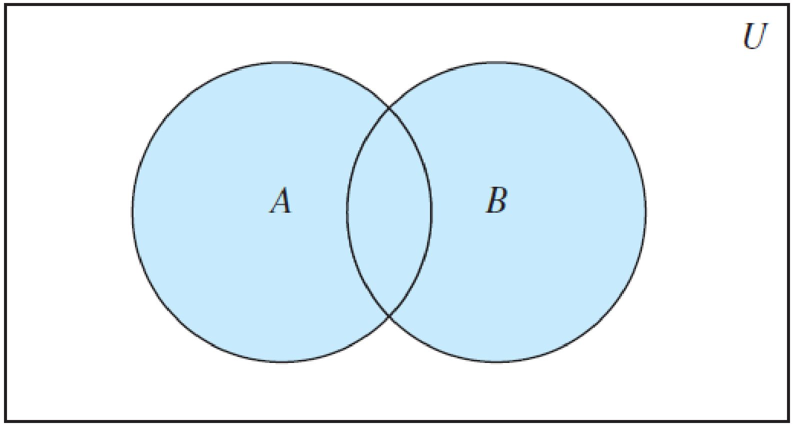
CARTESIAN PRODUCTS

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. The Cartesian product of $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$ while $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$. Therefore, $A \times B$ and $B \times A$ are not equal. The Cartesian product of more than two sets can also be defined, for example $A \times B \times A = \{(1, a, 1), (1, a, 2), (1, b, 1), (1, b, 2), (1, c, 1), (1, c, 2), (2, a, 1), (2, a, 2), (2, b, 1), (2, b, 2), (2, c, 1), (2, c, 2)\}$.

SET OPERATIONS

UNION

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. This tells us that $A \cup B = \{x \mid x \in A \lor x \in B\}$. Example: The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}.

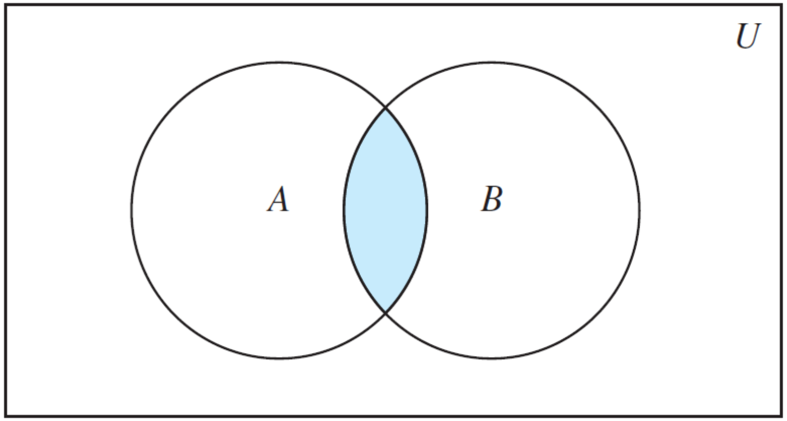


A ∪ *B* is shaded.

FIGURE 3 Venn Diagram of the Union of *A* and *B*

INTERSECTION

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B. Example: The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}. This tells us that $A \cap B = \{x \mid x \in A \land x \in B\}$. Note that two sets are called disjoint if their intersection is the empty set.

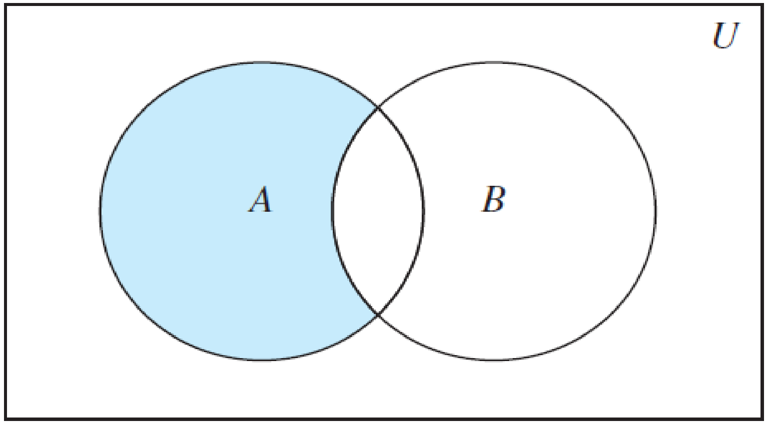


$A \cap B$ is shaded.

FIGURE 4 Venn Diagram of the Intersection of A and B

DIFFERENCE

Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A. This tells us that $A - B = \{x \mid x \in A \land x \notin B\}$. Take note that the difference of sets A and B is sometimes denoted by A\B.

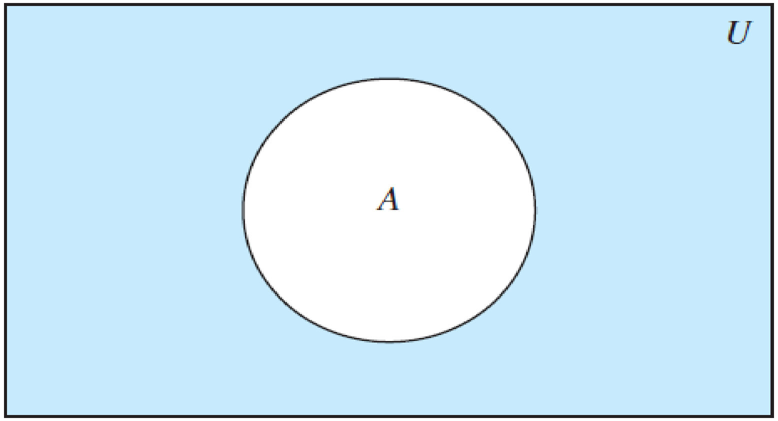


A - *B* is shaded.

FIGURE 5 Venn Diagram for the Difference of A and B

COMPLEMENT

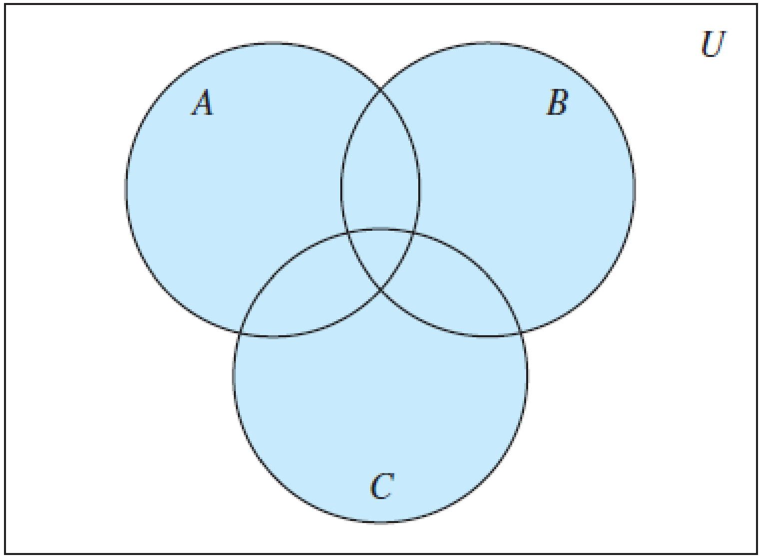
Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is U − A. This tells us that $\overline{A} = \{x \in U \mid x \notin A\}$.



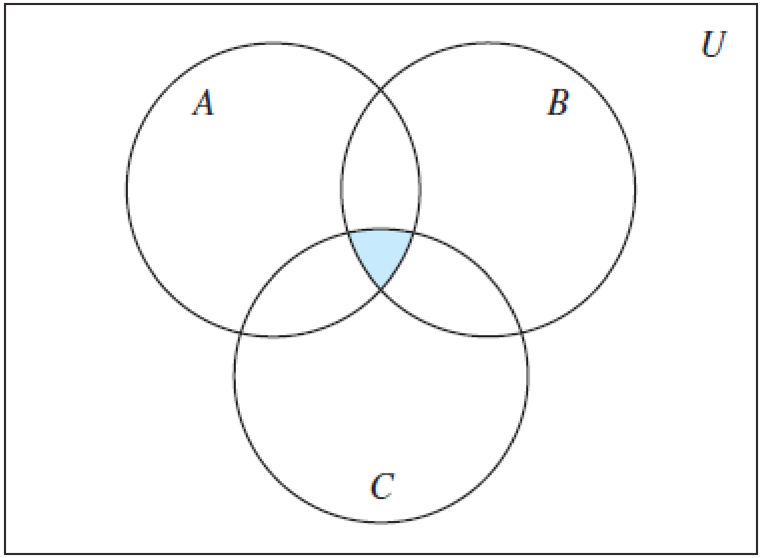
$\overline{A}$ shaded.

FIGURE 6 Venn Diagram for the Complement of the Set *A*

MORE EXAMPLES



A U *B* U *C* is shaded.



A U *B* U *C* is shaded.

FIGURE 7 The Union and Intersection of *A*, *B*, and *C*

SET IDENTITIES

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

SET IDENTITIES

Identity	Name
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

FUNCTIONS

INTRODUCTION

In many instances, we assign to each element of a set a particular element of a second set. For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A,B,C,D,F}. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment is an example of a function.

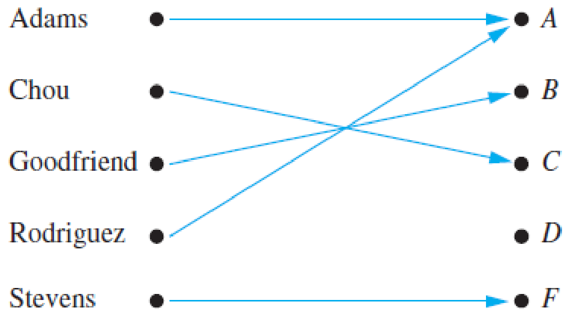


FIGURE 8 **Assignment of Grades in a Discrete Mathematics Class**

FUNCTION DEFINED

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$. Functions are sometimes also called mappings or transformations.

FUNCTION DEFINED

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If f (a) = b, we say that b is the image of a and a is a preimage of b. The range, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

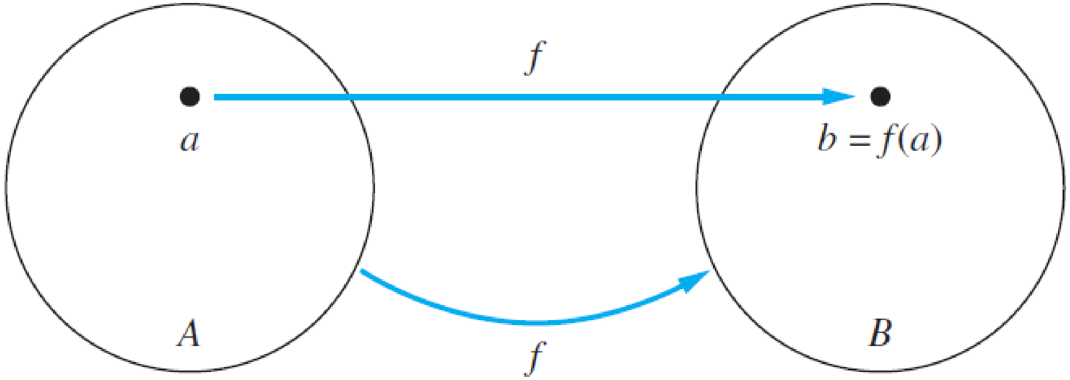


FIGURE 9 **The Function *f* Maps *A* to *B***

ONE-TO-ONE FUNCTION

A function f is said to be one-to-one, or an injunction, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

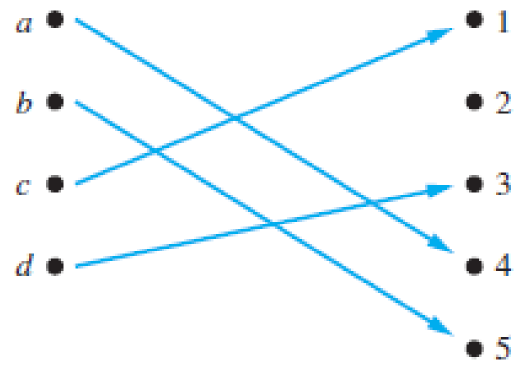


FIGURE 10 A One-to-One Function

ONTO FUNCTION

A function f from A to B is called onto, or a surjection, if and only if for every element b ∈ B there is an element a ∈ A with f (a) = b. A function f is called surjective if it is onto.

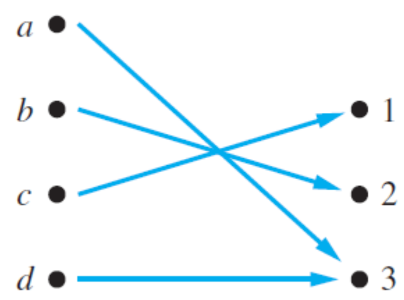


FIGURE 11 **An Onto Function**

DIFFERENT TYPES OF CORRESPONDENCES

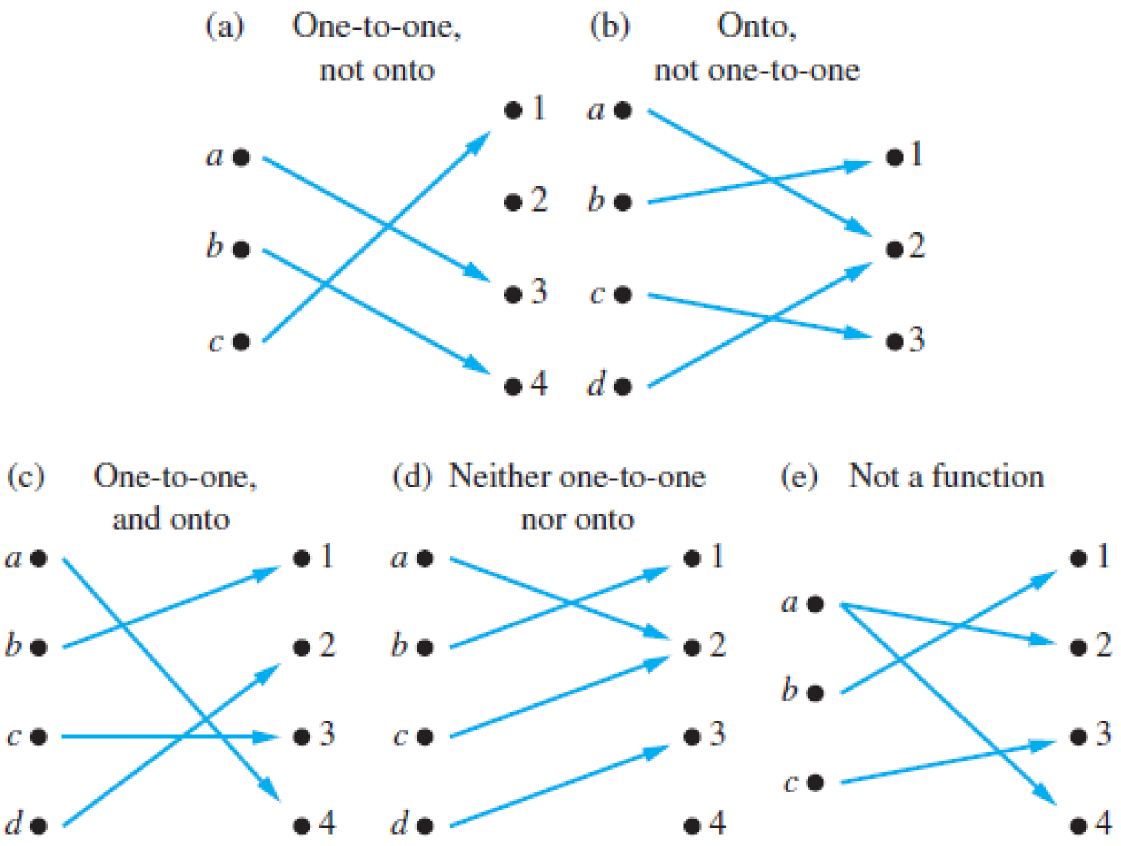


FIGURE 12 **Examples of Different Types of Correspondences**

SEQUENCES

INTRODUCTION

Sequences are ordered lists of elements, used in discrete mathematics in many ways. For example, they can be used to represent solutions to certain counting problems. They are also an important data structure in computer science. We will often need to work with sums of terms of sequences in our study of discrete mathematics.

SEQUENCE DEFINED

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81 , . . . , 3n, . . . is an infinite sequence.

A sequence is a function from a subset of the set of integers to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

TYPES OF PROGRESSION

1. A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

   where the initial term a and the common ratio r

   are real numbers.

Example: a=1, r=3;

Sequence: 1, 3, 9, 27, 81, 243, 729, …

TYPES OF PROGRESSION

2. An arithmetic progression is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Example: $a=-4$, $d=2$;

Sequence: -4, -2, 0, 2, 4, 6, 8, 10, 12, 14, …

FIBONACCI SEQUENCE

The Fibonacci sequence, f0, f1, f2, . . . , is defined by the initial conditions f0 = 0, f1 = 1, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for n = 2, 3, 4, ...

$f_2 = f_1 + f_0 = 1 + 0 = 1$,

$f_3 = f_2 + f_1 = 1 + 1 = 2$,

$f_4 = f_3 + f_2 = 2 + 1 = 3$,

$f_5 = f_4 + f_3 = 3 + 2 = 5$,

$f_6 = f_5 + f_4 = 5 + 3 = 8$.

Sequence: 0, 1, 1, 3, 5, 8, 13, 21, 34, 55, 89, …

SUMMATIONS

SUMMATION

Summation is the addition of the terms of a sequence. We use the summation notation $\sum_{j=m}^{n} a_j$ read as the sum from j = m to j = n of a_j to represent $a_m + a_{m+1} + \cdot \cdot \cdot + a_n$. Here, the variable j is called the index of summation, and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k. Here, the index of summation runs through all integers starting with its lower limit m and ending with its upper limit n. A large uppercase Greek letter sigma,∑, is used to denote summation.

EXAMPLE

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2\\ &= 1 + 4 + 9 + 16 + 25\\ &= 55\end{aligned}$$

$$\begin{aligned}\sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8\\ &= 1 + (-1) + 1 + (-1) + 1\\ &= 1\end{aligned}$$

MATRICES

MATRIX DEFINED

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3 x 2 matrix.

MATRIX DEFINED

Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The ith row of A is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The jth column of A is the $m \times 1$ matrix.

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}.$$

MATRIX ARITHMETIC

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The sum of A and B, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element. In other words, $A + B = [a_{ij} + b_{ij}]$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

MATRIX ARITHMETIC

Let A be an $m \times k$ matrix and B be a $k \times n$ matrix. The product of A and B, denoted by AB, is the $m \times n$ matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of A and the jth column of B. In other words, if $AB = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

END OF CHAPTER 2