Quantum Mechanics

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Linear Equations
with constant coefficients

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Introduction

Let us consider a differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where the coefficients a_k are constants.

This equation is **linear** in y. The solution to this type of equation can be drawn from the fact that the exponential function is the only function whose derivative is itself, and

$$\frac{de^{kx}}{dx} = ke^x$$

$$\frac{d^2 e^{kx}}{dx^2} = k^2 e^x; \quad \frac{d^3 e^{kx}}{dx^3} = k^3 e^x; \quad \cdots \quad ; \frac{d^n e^{kx}}{dx^n} = k^n e^x$$

Auxiliary Equation

If we let

$$y = e^{kx}$$

then the differential equation yields

$$a_0 k^n e^{kx} + a_1 k^{n-1} e^{kx} + \cdots + a_{n-1} k e^{kx} + a_n e^{kx} = 0$$

which is an algebraic equation where e^{kx} could be factored out. We then have

$$a_0 k^n + a_1 k^{n-1} + \cdots + a_{n-1} k + a_n = 0$$

This n-degree polynomial function in k is called the **auxiliary equation** associated with the differential equation.

General Solution

Let $\{m_i\}$ be the roots of the auxiliary equation. Then each of

$$y_i = e^{m_i x}$$

clearly satisfies the differential equation. As an n – order differential equation has an n – degree polynomial auxiliary equation, which has n roots, the differential has n possible solutions. Linear combinations of such solutions are also solutions of the differential equation.

Thus, the general solution of the differential equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + \ c_n \, e^{m_n x}$$

where $\{c_i\}$ are the constants of integration.

Examples

<u>Example 1</u>

$$3\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

Solution:

The auxiliary equation is

$$3k^3 + 5k^2 - 2k = 0$$

which may be written as

$$k(k+2)(k-1/3) = 0$$

The roots of the auxiliary equation are then $k = 0, -2, 1/3$. The general solution of the differential equation is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x/3}$$

Examples

Example 2

$$\frac{d^2x}{dt^2} - 4x = 0; \qquad \text{when } t = 0, x = 0, \frac{\text{dx}}{dt} = 3$$

Solution:

The auxiliary equation is

$$k^2 - 4 = 0$$

The roots of the auxiliary equation are then $k = 2, -2$. The general solution of the differential equation is

$$x = c_1 e^{2t} + c_2 e^{-2t}$$

Examples

Example 2 (cont'd)

Applying the initial conditions,

$$x(0) = 0 = c_1 + c_2$$

The first derivative of x being

$$\frac{dx}{dt} = 2c_1e^{2t} - 2c_2e^{-2t}$$

$$\frac{dx}{dt}(0) = 3 = 2c_1 - 2c_2$$

We then find that

$$c_1 = \frac{3}{4}; \quad c_2 = -\frac{3}{4}$$

Thus,

$$x = \frac{3}{4}(e^{2t} - e^{-2t}) = \frac{3}{2}\sinh 2t$$

Examples

<u>Example 3</u>

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + 13x = 0$$

Solution:

The auxiliary equation is

$$k^3 - 3k^2 + 9k + 13 = 0$$

which may be recast as

$$(k + 1)(k^2 - 4k + 13) = 0$$

The roots of the auxiliary equation are then $k = -1, 2 + 3i, 2 - 3i$. The general solution of the differential equation is

$$y = c_1e^{-x} + c_2e^{(2+3i)x} + c_3e^{(2-3i)x}$$
$$= c_1e^{-x} + \left(c_2e^{i3x} + c_3e^{-i3x}\right)e^{2x}$$