

Quantum Mechanics

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Time-Dependent Perturbation Theory



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Time Evolution Operator

As there are few time-dependent potentials that have exact solutions, let us consider a perturbation theory for these types of potentials. We start by looking at the time evolution of the wave function in the interaction picture.

Let

$$\psi_I(x, t) = U_I(t, t_0)\psi(x, t_0)$$

where for consistency the interaction-picture time evolution operator satisfies

$$U_I(t_0, t_0) = 1$$

In the interaction picture,

$$i\hbar \frac{\partial \psi_I(x, t)}{\partial t} = V_I \psi_I(x, t)$$

The time-evolution operator therefore satisfies

$$i\hbar \frac{dU_I(t, t_0)}{dt} = V_I U_I(t, t_0)$$



Dyson Series

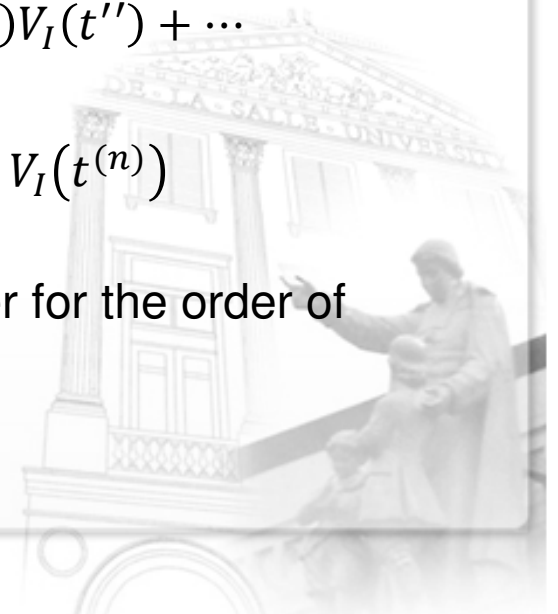
The differential equation may be inverted, giving

$$U_I(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t V_I(t') U_I(t', t_0) dt'$$

Iterating,

$$\begin{aligned} U_I(t, t_0) &= 1 - \frac{i}{\hbar} \int_{t_0}^t V_I(t') \left[1 - \frac{i}{\hbar} \int_{t_0}^{t'} V_I(t'') U_I(t'', t_0) dt'' \right] dt' \\ &= 1 + \left(-\frac{i}{\hbar} \right) \int_{t_0}^t V_I(t') dt' + \left(-\frac{i}{\hbar} \right)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V_I(t') V_I(t'') + \dots \\ &+ \left(-\frac{i}{\hbar} \right)^n \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \dots \int_{t_0}^{t^{(n-1)}} dt^{(n)} V_I(t') V_I(t'') \dots V_I(t^{(n)}) \end{aligned}$$

This is the Dyson series. The factor $-i/\hbar$ serves as a marker for the order of perturbation.



Expansion Coefficients

Suppose the system is initially in an eigenstate $|i\rangle$ of H_0 . Then we may write

$$|\psi(t_0)\rangle_I = |i\rangle$$

At a later time,

$$|\psi(t)\rangle_I = U_I(t, t_0)|i\rangle$$

Using the completeness relation, we may recast this as

$$|\psi(t)\rangle_I = \sum_n |n\rangle \langle n| U_I(t, t_0) |i\rangle$$

Identifying with the expansion postulate

$$|\psi(t)\rangle_I = \sum_n c_n(t) |n\rangle$$

we have

$$c_n(t) = \langle n| U_I(t, t_0) |i\rangle$$



Perturbation Terms

If we expand the coefficients as perturbative terms

$$c_n(t) = c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + \dots$$

and insert the Dyson series in

$$c_n(t) = \langle n | U_I(t, t_0) | i \rangle$$

We have

$$c_n^{(0)} = \delta_{ni}$$

$$c_n^{(1)} = \left(-\frac{i}{\hbar}\right) \int_{t_0}^t \langle n | V_I(t') | i \rangle dt' = \left(-\frac{i}{\hbar}\right) \int_{t_0}^t V_{ni}(t') e^{i\omega_{ni}t'} dt'$$

$$c_n^{(2)} = \left(-\frac{i}{\hbar}\right)^2 \sum_m \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' V_{nm}(t') V_{mi}(t'') e^{i\omega_{nm}t'} e^{i\omega_{mi}t''}$$

