# Quantum Mechanics

*Robert C. Roleda*
*Physics Department*

## Time-Dependent Perturbation Theory

De La Salle University

# Time Evolution Operator

As there are few time-dependent potentials that have exact solutions, let us consider a perturbation theory for these types of potentials. We start by looking at the time evolution of the wave function in the interaction picture.

Let

$$\psi_I(x,t) = U_I(t,t_0)\psi(x,t_0)$$

where for consistency the interaction-picture time evolution operator satisfies

$$U_I(t_0,t_0) = 1$$

In the interaction picture,

$$i\hbar\frac{\partial\psi_I(x,t)}{\partial t} = V_I\psi_I(x,t)$$

The time-evolution operator therefore satisfies

$$i\hbar\frac{dU_I(t,t_0)}{dt} = V_I U_I(t,t_0)$$

# Dyson Series

The differential equation may be inverted, giving

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t',t_0)dt'$$

Iterating,

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\left[1 - \frac{i}{\hbar}\int_{t_0}^{t'} V_I(t'')U_I(t'',t_0)dt''\right]dt'$$

$$= 1 + \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t} V_I(t')dt' + \left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\, V_I(t')V_I(t'') + \cdots$$

$$+ \left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)}\, V_I(t')V_I(t'')\cdots V_I\left(t^{(n)}\right)$$

This is the Dyson series. The factor $-i/\hbar$ serves as a marker for the order of perturbation.

# Expansion Coefficients

Suppose the system is initially in an eigenstate $|i\rangle$ of $H_0$. Then we may write

$$|\psi(t_0)\rangle_I = |i\rangle$$

At a later time,

$$|\psi(t)\rangle_I = U_I(t, t_0)|i\rangle$$

Using the completeness relation, we may recast this as

$$|\psi(t)\rangle_I = \sum_n |n\rangle\langle n|U_I(t, t_0)|i\rangle$$

Identifying with the expansion postulate

$$|\psi(t)\rangle_I = \sum_n c_n(t)|n\rangle$$

we have

$$c_n(t) = \langle n|U_I(t, t_0)|i\rangle$$

# Perturbation Terms

If we expand the coefficients as perturbative terms

$$c_n(t) = c_n^{(0)} + c_n^{(1)} + c_n^{(2)} + \cdots$$

and insert the Dyson series in

$$c_n(t) = \langle n|U_I(t, t_0)|i\rangle$$

We have

$$c_n^{(0)} = \delta_{ni}$$

$$c_n^{(1)} = \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t}V_{ni}(t')e^{i\omega_{ni}t'}dt'$$

$$c_n^{(2)} = \left(-\frac{i}{\hbar}\right)^2\sum_m \int_{t_0}^{t}dt' \int_{t_0}^{t'}dt''\, V_{nm}(t')V_{mi}(t'')e^{i\omega_{nm}t'}e^{i\omega_{mi}t''}$$