Quantum Mechanics

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Separation of Variables

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Separable Equations of Order-One

The simplest ODE of order-one have the following form

$$P(x)dx + Q(y)dy = 0$$

The solution is found by simply integrating.

Example 1

$$\frac{dy}{dx} = -y^2 e^x$$

can be rewritten as

$$-\frac{dy}{y^2} = e^x dx$$

Each side can be integrated separately. Thus

$$\boxed{\frac{1}{y} = e^x + c}$$

General and Particular Solutions

Example 2

$$2x(y+1)dx - ydy = 0$$

where $y = -2$ when $x = 0$.

Solution:

Put x and y in separate terms,

$$2xdx - \frac{ydy}{y+1} = 0$$

This can be recast for easier integration

$$2xdx = \left(1 - \frac{1}{y+1}\right)dy$$

$$1 - \frac{1}{y+1} = \frac{y+1-1}{y+1} = \frac{y}{y+1}$$

Integrating each side yields the general solution

$$x^2 = y - ln|y+1| + c$$

General and Particular Solutions

Example 2 (cont'd)

Applying the initial conditions y = -2 when x = 0,

$$0 = -2 - ln|-2+1| + c$$

indicating that

$$c = 2 + ln1 = 2$$

0

The particular solution is therefore

$$\boxed{x^2 = y - ln|y+1| + 2}$$

Change of Variables

Sometimes when the variables are not separable, a differential equation may be reduced to one by an appropriate change of variable. The general form of differential equation that is amenable to this approach is

$$\frac{dy}{dx} = f(ax + by)$$

If we let

$$w = ax + by$$

then

$$\frac{dw}{dx} = a + b\frac{dy}{dx}$$

and the differential equation can be rewritten as

$$\frac{dw}{dx} - a = bf(w)$$

which is separable as

$$\frac{dw}{a + bf} = dx$$

Change of Variables

Example 3

$$\frac{dy}{dx} = 8x + 4y + (2x + y - 1)^2$$

Solution:

Let

$$w = 2x + y$$

Then

$$\frac{dw}{dx} = 2 + \frac{dy}{dx}$$

The differential equation can then be recast as

$$\frac{dw}{dx} - 2 = 4w + (w - 1)^2$$

Separating the variables

$$\frac{dw}{[4w + (w - 1)^2 + 2]} = dx$$

Change of Variables

Example 3 (cont'd)

The denominator on the left can be simplified

$$\frac{dw}{[w^2 + 2w + 3]} = dx$$

$$4w + (w-1)^2 + 2$$
$$= 4w + w^2 - 2w + 1 + 2$$
$$= w^2 + 2w + 3$$

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}$$

Integrating each side,

$$\frac{1}{\sqrt{8}} \arctan \frac{2w + 2}{\sqrt{8}} = x + c$$

This can be recast as

$$\frac{w + 1}{\sqrt{2}} = \tan \sqrt{8}(x + c)$$

or

$$y = \sqrt{2} \tan \sqrt{8}(x + c) - 2x - 1$$