Quantum Mechanics

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Two-Level Systems

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Two-Level Systems

- There are few exact solutions for time-dependent potentials
- Two-level systems are an exception: nuclear magnetic resonance, laser, maser

Let

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$$

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$$

with γ, ω real positive constants, and $E_1 < E_2$.

Without losing generality, let us consider the case where initially only the lower state is populated

$$c_1(0) = 1. \quad c_2(0) = 0$$

From [interaction picture],the probability amplitude of each state evolves as follows

$$i\hbar \frac{dc_n}{dt} = \sum_m V_{nm} e^{i\omega_{nm} t} c_m$$

Evolution of Amplitudes

We note that

$$V_{11} = 0 \quad V_{12} = \gamma e^{i\omega t} \quad V_{21} = \gamma e^{-i\omega t} \quad V_{22} = 0$$

Thus,

$$i\hbar \frac{dc_1}{dt} = \gamma e^{i\omega t} e^{i\omega_{12} t} c_2$$

$$i\hbar \frac{dc_2}{dt} = \gamma e^{-i\omega t} e^{i\omega_{21} t} c_1$$

where $E_n - E_m = \hbar\omega_{nm}$.

Taking the second derivative of the second equation,

$$i\hbar \frac{d^2 c_2}{dt^2} = \gamma e^{i(\omega_{21}-\omega)t} \frac{dc_1}{dt} + i(\omega_{21} - \omega)\gamma e^{i(\omega_{21}-\omega)t} c_1$$

Referring to the two original equations, we can recast the second derivative as

$$i\hbar \frac{d^2 c_2}{dt^2} = \frac{\gamma^2}{i\hbar} c_2 + +i(\omega_{21} - \omega) i\hbar \frac{dc_2}{dt}$$

General Solution

The differential equation can be reorganized as

$$\frac{d^2c_2}{dt^2} - i(\omega_{21} - \omega)\frac{dc_2}{dt} + \frac{\gamma^2}{\hbar^2}c_2 = 0$$

This is a second-order differential equation with constant coefficients. For the ansatz $c_2 = e^{kt}$, the auxiliary equation is

$$k^2 - i(\omega_{21} - \omega)k + \frac{\gamma^2}{\hbar^2} = 0$$

which has the roots

$$k = \frac{i(\omega_{21} - \omega) \pm \sqrt{-(\omega_{21} - \omega)^2 - 4(\gamma/\hbar)^2}}{2} = i\frac{(\omega_{21} - \omega)}{2} \pm i\sqrt{\frac{(\omega_{21} - \omega)^2}{4} + \frac{\gamma^2}{\hbar^2}}$$

The general solution is then

$$c_2(t) = e^{i(\omega_{21} - \omega)t/2}[A \sin \Omega t + B \cos \Omega t]$$

where

$$\Omega = \sqrt{\frac{(\omega_{21} - \omega)^2}{4} + \frac{\gamma^2}{\hbar^2}}$$

Initial Conditions

Applying the initial condition $c_2(0) = 0$, we have

$$B = 0$$

Thus,

$$c_2(t) = Ae^{i(\omega_{21}-\omega)t/2}\sin\Omega t$$

$$\frac{dc_2}{dt} = \frac{i(\omega_{21}-\omega)}{2}Ae^{i(\omega_{21}-\omega)t/2}\sin\Omega t + \Omega Ae^{i(\omega_{21}-\omega)t/2}\cos\Omega t$$

Comparing with

$$i\hbar\frac{dc_2}{dt} = \gamma e^{i(\omega_{21}-\omega)t}c_1$$

we have

$$c_1(t) = \frac{i\hbar}{\gamma}Ae^{-i(\omega_{21}-\omega)t/2}[\sin\Omega t + \Omega\cos\Omega t]$$

With $c_1(0) = 1$

$$A = \frac{\gamma}{i\hbar\Omega}$$

Initial Conditions

We then have

$$c_1(t) = e^{-i(\omega_{21}-\omega)t/2}\left[\frac{1}{\Omega}\sin\Omega t + \cos\Omega t\right]$$

$$c_2(t) = \frac{\gamma}{i\hbar\Omega} e^{i(\omega_{21}-\omega)t/2}\sin\Omega t$$

If the system is initially in the lower energy state, the probability that at a later time t, it is found in the higher energy state is

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega_{21}-\omega)^2/4}\sin^2\left(\sqrt{\frac{(\omega_{21}-\omega)^2}{4} + \frac{\gamma^2}{\hbar^2}}\ t\right)$$

This is the Rabi formula, named after the father of molecular beam Isaac Rabi.

The probability that it remains in the lower state is

$$|c_1(t)|^2 = 1 - |c_2(t)|^2$$