Quantum Mechanics 2

Robert C. Roleda
Physics Department

Perturbation Theory

Damped Oscillator

De La Salle University

Damped Oscillator

Let us consider a damped oscillator for which the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + bx$$

We know the exact solutions for the Harmonic oscillator. We may thus assign

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$H_I = bx$$

The eigenenergies of a Harmonic oscillator are

$$E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar\omega$$

Ladder Operators

We recall from [ladder 4] that the position operator may be expressed in terms of ladder operators

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(A + A^\dagger\right)$$

and the ladder operators act on the harmonic oscillator energy eigenkets in the following ways:

$$A\left|n^{(0)}\right\rangle = \sqrt{n}\,\left|(n-1)^{(0)}\right\rangle$$

$$A^\dagger\left|n^{(0)}\right\rangle = \sqrt{n+1}\,\left|(n+1)^{(0)}\right\rangle$$

First-Order Energy

The first-order correction for energy is

$$E_n^{(1)} = \langle n^{(0)} | H_I | n^{(0)} \rangle = \langle n^{(0)} | bx | n^{(0)} \rangle = \sqrt{\frac{\hbar b^2}{2m\omega}} \langle n^{(0)} | (A + A^\dagger) | n^{(0)} \rangle$$

Now,

$$\langle n^{(0)} | A | n^{(0)} \rangle = \sqrt{n} \langle n^{(0)} | (n-1)^{(0)} \rangle = 0$$

$$\langle n^{(0)} | A^\dagger | n^{(0)} \rangle = \sqrt{n+1} \langle n^{(0)} | (n+1)^{(0)} \rangle = 0$$

Hence,

$$E_n^{(1)} = 0$$

Second-Order Energy

The second-order correction to the energy is

$$E_n^{(2)} = \sum_{k\neq n} \frac{\left|\langle n^{(0)}|H_I|k^{(0)}\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} = \frac{\hbar b^2}{2m\omega} \sum_{k\neq n} \frac{\left|\langle n^{(0)}|(A + A^\dagger)|k^{(0)}\rangle\right|^2}{(n-k)\hbar\omega}$$

Since

$$\langle n^{(0)}|A|k^{(0)}\rangle = \sqrt{k}\langle n^{(0)}|(k-1)^{(0)}\rangle = \sqrt{k}\delta_{n,k-1}$$

$$\langle n^{(0)}|A^\dagger|k^{(0)}\rangle = \sqrt{k+1}\langle n^{(0)}|(k+1)^{(0)}\rangle = \sqrt{k+1}\delta_{n,k+1}$$

$$\left|\langle n^{(0)}|(A + A^\dagger)|k^{(0)}\rangle\right|^2 = k\delta_{n,k-1}\delta_{n,k-1} + 2\sqrt{k}\sqrt{k+1}\delta_{n,k-1}\delta_{n,k+1} + (k+1)\delta_{n,k+1}\delta_{n,k+1}$$

And summing over k,

$$E_n^{(2)} = \frac{\hbar b^2}{2m\omega}\left[\frac{(n+1)\delta_{nn}}{-\hbar\omega} + \frac{2\sqrt{n(n-1)}\delta_{n,n-2}}{\hbar\omega} + \frac{n\delta_{nn}}{\hbar\omega}\right] = \frac{\hbar b^2}{2m\omega}\left[-\frac{(n+1)}{\hbar\omega} + \frac{n}{\hbar\omega}\right]$$

The lowest-order correction to the energy is then

$$E_n^{(2)} = -\frac{b^2}{2m\omega^2}$$

First-Order Eigenkets

The first order coefficient for the eigenkets are

$$C_{nk}^{(1)} = \frac{\langle k^{(0)}|H_I|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}} = \sqrt{\frac{\hbar b^2}{2m\omega}} \frac{\langle k^{(0)}|(A + A^\dagger)|n^{(0)}\rangle}{(n-k)\hbar\omega} = \sqrt{\frac{\hbar b^2}{2m\omega}} \frac{\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}}{(n-k)\hbar\omega}$$

To first-order, the energy eigenkets are

$$|n^{(1)}\rangle = N\left[|n^{(0)}\rangle + \sum_{k\neq n} C_{nk}^{(1)} |k^{(0)}\rangle\right]$$

Hence,

$$|n^{(1)}\rangle = N\left[|n^{(0)}\rangle + \sqrt{\frac{nb^2}{2m\hbar\omega^3}}|(n-1)^{(0)}\rangle - \sqrt{\frac{(n+1)b^2}{2m\hbar\omega^3}}|(n+1)^{(0)}\rangle\right]$$

The perturbed ground state is for example

$$|0^{(1)}\rangle = N\left[|0^{(0)}\rangle - \sqrt{\frac{b^2}{2m\hbar\omega^3}}|1^{(0)}\rangle\right]$$

Renormalization

We note that even if the harmonic oscillator eigenstates $|n^{(0)}\rangle$ are normalized, the perturbed eigenstates are linear combinations of $|n^{(0)}\rangle$, and these have to be normalized again

$$|n^{(1)}\rangle = N\left[|n^{(0)}\rangle + \sqrt{\frac{nb^2}{2m\hbar\omega^3}}\,|(n-1)^{(0)}\rangle - \sqrt{\frac{(n+1)b^2}{2m\hbar\omega^3}}\,|(n+1)^{(0)}\rangle\right]$$

The renormalization constant is

$$N = \left[1 + \frac{nb^2}{2m\hbar\omega^3} + \frac{(n+1)b^2}{2m\hbar\omega^3}\right]^{-1/2} = \left[1 + \frac{(2n+1)b^2}{2m\hbar\omega^3}\right]^{-1/2}$$

Renormalization

Thus the renormalized perturbed ground state is

$$|0^{(1)}\rangle = \left[1 + \frac{b^2}{2m\hbar\omega^3}\right]^{-1/2} \left[|0^{(0)}\rangle - \sqrt{\frac{b^2}{2m\hbar\omega^3}}\,|1^{(0)}\rangle\right]$$

the first excited state is

$$|1^{(1)}\rangle = \left[1 + \frac{3b^2}{2m\hbar\omega^3}\right]^{-1/2} \left[\sqrt{\frac{b^2}{2m\hbar\omega^3}}\,|0^{(0)}\rangle + |1^{(0)}\rangle - \sqrt{\frac{2b^2}{2m\hbar\omega^3}}\,|2^{(0)}\rangle\right]$$

and the second excited state is

$$|2^{(1)}\rangle = \left[1 + \frac{5b^2}{2m\hbar\omega^3}\right]^{-1/2} \left[\sqrt{\frac{2b^2}{2m\hbar\omega^3}}\,|1^{(0)}\rangle + |2^{(0)}\rangle - \sqrt{\frac{3b^2}{2m\hbar\omega^3}}\,|3^{(0)}\rangle\right]$$

Exact Solution

The damped oscillator is actually an exactly solvable problem. The potential term is quadratic

$$V = \frac{1}{2} m\omega^2 x^2 + bx$$

By completing the square,

$$V = \frac{1}{2} m\omega^2 \left[x^2 + \frac{2b}{m\omega^2} x + \left(\frac{b}{m\omega^2} \right)^2 - \left(\frac{b}{m\omega^2} \right)^2 \right] = \frac{1}{2} m\omega^2 \left(x + \frac{b}{m\omega^2} \right)^2 - \frac{b^2}{2m\omega^2}$$

and changing the variable

$$x \to y = x + \frac{b}{m\omega^2}$$

The Schrödinger equation may be recast as

$$-\frac{\hbar^2}{2m} \frac{\partial^2 u(y)}{\partial y^2} + \frac{1}{2} m\omega^2 y^2 u(y) = \left(E_n + \frac{b^2}{2m\omega^2} \right) u(y)$$

Exact Solution

The eigenvalues of this differential equation are

$$\left(n+\frac{1}{2}\right)\hbar\omega$$

Thus, the eigenenergies of the damped oscillator are

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{b^2}{2m\omega^2}$$

Note that the extra term on the right are exactly the second-order corrections calculated earlier.

Following [Harmonic Oscillator 5], the exact energy eigenstates of the damped oscillator are

$$u_n(x)=(2^n n!)^{-1/2}\left(\frac{m\omega}{\hbar}\right)^{1/4}H_n\left(\sqrt{\frac{m\omega}{\hbar}}\left(x+\frac{b}{m\omega^2}\right)\right)\exp\left(-\frac{m\omega}{2\hbar}\left(x+\frac{b}{m\omega^2}\right)^2\right)$$