Quantum Mechanics

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Gamma Functions

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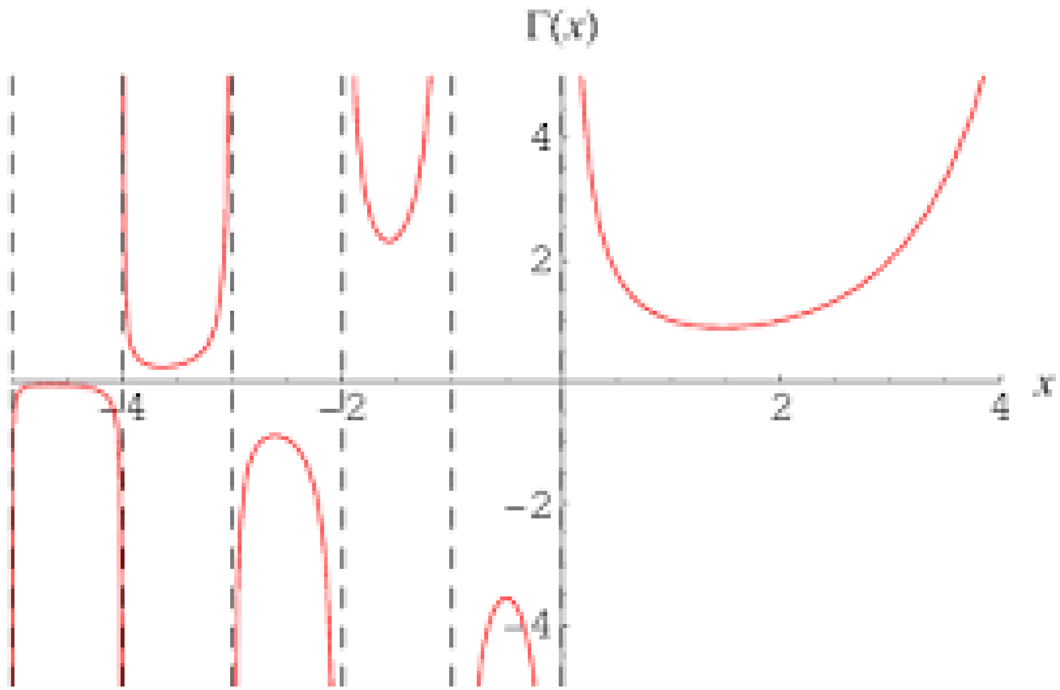
The Gamma Function

The Gamma Function is defined for $z > 0$, by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t} dt$$



Recursion Formulas

What is perhaps the most defining property of the Gamma Functions is its recursion formula.

$$\Gamma(z+1) = z\Gamma(z)$$

This property allows to analytically continue onto the $z < 0$ domain, so that for $1 - |z| > 0$,

$$\Gamma(-|z|) = -\frac{\Gamma(1-|z|)}{|z|}$$

Factorials

For $z = n$, where n is an integer, successive application of recursion formula gives

$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma(n) = (n-1)\Gamma(n-1)$$
$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$
$$\vdots$$
$$\Gamma(2) = 1 \cdot \Gamma(1)$$

Now,

$$\Gamma(1) = \int_0^\infty t^{1-1}\, e^{-t} dt = \int_0^\infty e^{-t} dt = [-e^{-t}]_0^\infty = -0 + 1 = 1$$

Thus,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)\cdots\Gamma(1) = n!$$

Negative Arguments

We now note that

$$\Gamma(1) = 0 \cdot \Gamma(0)$$

Thus, $\Gamma(0)$ is undefined. For negative integers, $\Gamma(-n)$ are likewise undefined as

$$\Gamma(0) = -1 \cdot \Gamma(-1)$$

$$\Gamma(-1) = -2 \cdot \Gamma(-2)$$

and so on. However, gamma function is well-defined for negative z that are not integers, because $\Gamma(z)$ for $-1 < z < 0$ can be derived from

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

since $\Gamma(z+1)$ for $0 < z+1 < 1$ is well-defined. Likewise, $\Gamma(z)$ for $-2 < z < -1$ can be derived from

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

since $\Gamma(z+1)$ for $-1 < z+1 < 0$ is well-defined, and so on.

Alternate Integral Form

The Gamma Function integral for $z > 0$ may be cast in an alternate form if we let $t = u^2$. Hence

$$\Gamma(z) = \int_0^\infty u^{2(z-1)}\, e^{-u^2} 2u du = 2 \int_0^\infty u^{2z-1}\, e^{-u^2} du$$

Note that if $2z - 1 = n$, where n is a positive integer, $z = (n+1)/2$ are half-integers. Gamma of half-integers may also be evaluated using the recursion formula if we note that

$$\Gamma\left(\tfrac{1}{2}\right) = 2 \int_0^\infty e^{-u^2} du = \sqrt{\pi}$$

This is just the Gaussian Integral formula [see Gaussian Integral]