Quantum Mechanics 2

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Clebsch-Gordan Coefficients

$j_1 = 1, j_2 = 1$

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Quantum States

For $j_1 = 1, j_2 = 1$, the possible values of the quantum number j of the total angular momentum $J = J_1 + J_2$ are

$$j = 0, 1, 2$$

The $j = 0$ state is a singlet, the $j = 1$ states form a triplet, and the $j = 2$ states form a quintuplet.

The Clebsch Gordan coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ are defined through

$$|j_1 j_2 j m\rangle = \sum_{m_1}^{j_1} \sum_{m_2}^{j_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

Dispensing with the writing of the quantum numbers $j_1 j_2$, the coupled and uncoupled states are

$$|jm\rangle = |2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle, |2,-2\rangle, |1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle$$

$$|m_1 m_2\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,1\rangle, |0,0\rangle, |0,-1\rangle, |-1,1\rangle, |-1,0\rangle, |-1,-1\rangle$$

$j = 2$ States

We begin by taking note that from the addition of the $z-$ components,

$$m = m_1 + m_2$$

We then start from the "highest" $|jm\rangle$ state $|2{,}2\rangle$. The only $|m_1 m_2\rangle$ state that will satisfy $m = m_1 + m_2$ is $|1{,}1\rangle$. Thus,

$$\boxed{|2{,}2\rangle' = |1{,}1\rangle}$$

where to differentiate between the two sets of kets, we denote $|jm\rangle$ states by a prime.

We now use the lowering operator

$$J_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\hbar|j,m-1\rangle$$

to evaluate the "lower" states, noting that

$$J_- = J_{1-} + J_{2-}$$

and that J_{i-} acts only on the m_i states.

$j = 2$ States

We also note that

$$j(j+1) - m(m-1) = j^2 + j - m^2 + m = (j+m)(j-m+1)$$

So we have a more convenient expression

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Thus

$$J_-|2,2\rangle' = (J_{1-} + J_{2-})|1,1\rangle$$

yields

$$\sqrt{4}\hbar|2,2\rangle' = \sqrt{2}\hbar|0,1\rangle + \sqrt{2}\hbar|1,0\rangle$$

which gives

$$|2,1\rangle' = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle$$

$j = 2$ States

Using the lowering operator again,

$$J_-|2,1\rangle' = (J_{1-} + J_{2-})\left[\frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle\right]$$

we have

$$\sqrt{6}\hbar|2,0\rangle' = \left[\sqrt{2}\hbar\frac{1}{\sqrt{2}}|-1,1\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{2}}|0,0\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{2}}|0,0\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{2}}|1,-1\rangle\right]$$

which gives

$$|2,0\rangle' = \left[\frac{1}{\sqrt{6}}|-1,1\rangle + \frac{2}{\sqrt{6}}|0,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle\right]$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

$j = 2$ States

And once more,

$$J_-|2,0\rangle' = (J_{1-} + J_{2-})\left[\frac{1}{\sqrt{6}}|-1,1\rangle + \frac{2}{\sqrt{6}}|0,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle\right]$$

we have

$$\sqrt{6}\hbar|2,-1\rangle' = \left[\sqrt{2}\hbar\frac{1}{\sqrt{6}}|-1,0\rangle + \sqrt{2}\hbar\frac{2}{\sqrt{6}}|-1,0\rangle + \sqrt{2}\hbar\frac{2}{\sqrt{6}}|0,-1\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{6}}|0,-1\rangle\right]$$

which gives

$$|2,-1\rangle' = \frac{1}{\sqrt{2}}|-1,0\rangle + \frac{1}{2}|0,-1\rangle$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

$j = 2$ States

And once more,

$$J_-|2,-1\rangle' = (J_{1-} + J_{2-})\left[\frac{1}{\sqrt{2}}|-1,0\rangle + \frac{1}{2}|0,-1\rangle\right]$$

we have

$$\sqrt{4}\hbar|2,-2\rangle' = \left[\sqrt{2}\hbar\frac{1}{\sqrt{2}}|-1,-1\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{2}}|-1,-1\rangle\right]$$

which gives

$$|2,-2\rangle' = |-1,-1\rangle$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Clebsch-Gordan Coefficients

We can pick out the Clebsch-Gordan coefficients from the relations between the $|jm\rangle$ and $|m_1 m_2\rangle$. For $j = 2$, we have

$$|2,2\rangle' = |1,1\rangle$$

$$|2,1\rangle' = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle$$

$$|2,0\rangle' = \frac{1}{\sqrt{6}}|-1,1\rangle + \frac{2}{\sqrt{6}}|0,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle$$

$$|2,-1\rangle' = \frac{1}{\sqrt{2}}|-1,0\rangle + \frac{1}{2}|0,-1\rangle$$

$$|2,-2\rangle' = |-1,-1\rangle$$

We tabulate this on the right

To declutter, we leave all non-essential zeroes as blanks

$J = 2$		m				
m_1	m_2	2	1	0	-1	-2
1	1	1				
1	0		$1/\sqrt{2}$			
1	-1			$1/\sqrt{6}$		
0	1		$1/\sqrt{2}$			
0	0			$2/\sqrt{6}$		
0	-1				$1/\sqrt{2}$	
-1	1			$1/\sqrt{6}$		
-1	0				$1/\sqrt{2}$	
-1	-1					1

$j = 1$ States

We cannot use ladder operators to go from one j state to a state with a different value of j.

We do know that eigenstates of Hermitian operators are orthogonal

$$\langle j'm'|jm\rangle = \delta_{jj'}\delta_{mm'}$$

To move from a $j = 2$ state to a $j = 1$ state, let us then consider the orthogonality

$$\langle 2, m|1, m\rangle = 0$$

$j = 1$ States

Let us take the "highest" $m -$ state for $j = 1$. Since $m = m_1 + m_2$, we may write

$$|1{,}1\rangle' = a|0{,}1\rangle + b|1{,}0\rangle$$

Then

$$\langle 2{,}1|1{,}1\rangle' = \left[\frac{1}{\sqrt{2}}\langle 0{,}1| + \frac{1}{\sqrt{2}}\langle 1{,}0|\right][a|0{,}1\rangle + b|1{,}0\rangle] = \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b = 0$$

indicating that

$$b = \ a$$

Thus,

$$|1{,}1\rangle' = a|0{,}1\rangle - \ a|1{,}0\rangle$$

Normalizing, we have

$$\boxed{|1{,}1\rangle' = \frac{1}{\sqrt{2}}|0{,}1\rangle - \frac{1}{\sqrt{2}}|1{,}0\rangle}$$

$j = 1$ States

Using the lowering operator,

$$J_-|1,1\rangle' = (J_{1-} + J_{2-})\left[\frac{1}{\sqrt{2}}|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle\right]$$

we have

$$\sqrt{2}\hbar|1,0\rangle' = \left[\sqrt{2}\hbar\frac{1}{\sqrt{2}}|-1,1\rangle + \sqrt{2}\hbar\frac{1}{\sqrt{2}}|0,0\rangle - \sqrt{2}\hbar\frac{1}{\sqrt{2}}|0,0\rangle - \sqrt{2}\hbar\frac{1}{\sqrt{2}}|1,-1\rangle\right]$$

which gives

$$\boxed{|1,0\rangle' = \left[\frac{1}{\sqrt{2}}|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle\right]}$$

$$\boxed{J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle}$$

$j = 1$ States

And again,

$$J_-|1,0\rangle' = (J_{1-} + J_{2-})\left[\frac{1}{\sqrt{2}}|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle\right]$$

we have

$$\sqrt{2}\hbar|1,-1\rangle' = \left[\sqrt{2}\hbar\frac{1}{\sqrt{2}}|-1,0\rangle - \sqrt{2}\hbar\frac{1}{\sqrt{2}}|0,-1\rangle\right]$$

which gives

$$|1,-1\rangle' = \left[\frac{1}{\sqrt{2}}|-1,0\rangle - \frac{1}{\sqrt{2}}|0,-1\rangle\right]$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Clebsch-Gordan Coefficients

For $j = 1$, we have

$$|1,1\rangle' = \frac{1}{\sqrt{2}}|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle$$

$$|1,0\rangle' = \frac{1}{\sqrt{2}}|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle$$

$$|1,-1\rangle' = \frac{1}{\sqrt{2}}|-1,0\rangle - \frac{1}{\sqrt{2}}|0,-1\rangle$$

We tabulate the Clebsch-Gordan coefficients on the right.

$J = 1$		m		
m_1	m_2	1	0	-1
1	1			
1	0	$-1/\sqrt{2}$		
1	-1		$-1/\sqrt{2}$	
0	1	$1/\sqrt{2}$		
0	0		0	
0	-1			$-1/\sqrt{2}$
-1	1		$1/\sqrt{2}$	
-1	0			$1/\sqrt{2}$
-1	-1			

To declutter, we leave all non-essential zeroes as blanks

The Singlet State

Let us take the $j = 0$ state. Since $m = m_1 + m_2$, we may write

$$|0,0\rangle' = c|1,-1\rangle + d|0,0\rangle + e|-1,1\rangle$$

Then

$$\langle 1,0|0,0\rangle' = \left[-\frac{1}{\sqrt{2}}\langle 1,-1| + \frac{1}{\sqrt{2}}\langle -1,1|\right][c|1,-1\rangle + d|0,0\rangle + e|-1,1\rangle]$$

$$= -\frac{1}{\sqrt{2}}c + \frac{1}{\sqrt{2}}e = 0$$

indicating that $c = e$. On the other hand

$$\langle 2,0|0,0\rangle' = \left[\frac{1}{\sqrt{6}}\langle 1,-1| + \frac{2}{\sqrt{6}}\langle 0,0| + \frac{1}{\sqrt{6}}\langle -1,1|\right][c|1,-1\rangle + d|0,0\rangle + c|-1,1\rangle]$$

$$= \frac{1}{\sqrt{6}}c + \frac{2}{\sqrt{6}}d + \frac{1}{\sqrt{6}}c = 0$$

indicating that $d = -c$

Thus,

$$|0,0\rangle' = \frac{1}{\sqrt{3}}|1,-1\rangle - \frac{1}{\sqrt{3}}|0,0\rangle + \frac{1}{\sqrt{3}}|-1,1\rangle$$

Clebsch-Gordan Table

$j_1 = 1$ $j_2 = 1$		$J = 2$					$J = 1$			$J = 0$
		m								
m_1	m_2	2	1	0	-1	-2	1	0	-1	0
1	1	1								
1	0		$1/\sqrt{2}$				$-1/\sqrt{2}$			
1	-1			$1/\sqrt{6}$				$-1/\sqrt{2}$		$1/\sqrt{3}$
0	1		$1/\sqrt{2}$				$1/\sqrt{2}$			
0	0			$2/\sqrt{6}$				0		$-1/\sqrt{3}$
0	-1				$1/\sqrt{2}$				$-1/\sqrt{2}$	
-1	1			$1/\sqrt{6}$				$1/\sqrt{2}$		$1/\sqrt{3}$
-1	0				$1/\sqrt{2}$				$1/\sqrt{2}$	
-1	-1					1				