Quantum Mechanics 2

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Clebsch-Gordan Coefficients

$j_1 = 1, j_2 = \frac{1}{2}$

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Quantum States

For $j_1 = 1, j_2 = \frac{1}{2}$, the possible values of the quantum number j of the total angular momentum $J = J_1 + J_2$ are

$$j = \frac{1}{2}, \frac{3}{2}$$

The $j = \frac{1}{2}$ states form a doublet, the $j = \frac{3}{2}$ states form a quadruplet.

The Clebsch Gordan coefficients $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$ are defined through

$$|j_1 j_2 j m\rangle = \sum_{m_1}^{j_1} \sum_{m_2}^{j_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

Dispensing with the writing of the quantum numbers $j_1 j_2$, the coupled and uncoupled states are

$$|jm\rangle = \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle, \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle, \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle, \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle, \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$|m_1 m_2\rangle = \left|1, \tfrac{1}{2}\right\rangle, \left|1, -\tfrac{1}{2}\right\rangle, \left|0, \tfrac{1}{2}\right\rangle, \left|0, -\tfrac{1}{2}\right\rangle, \left|-1, \tfrac{1}{2}\right\rangle, \left|-1, -\tfrac{1}{2}\right\rangle$$

$j = 3/2$ States

We begin by taking note that from the addition of the $z-$ components,

$$m = m_1 + m_2$$

We then start from the "highest" $|jm\rangle$ state $\left|\frac{3}{2},\frac{3}{2}\right\rangle$. The only $|m_1 m_2\rangle$ state that will satisfy $m = m_1 + m_2$ is $\left|1,\frac{1}{2}\right\rangle$. Thus,

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle' = \left|1,\frac{1}{2}\right\rangle$$

where to differentiate between the two sets of kets, we denote $|jm\rangle$ states by a prime.

We now use the lowering operator

$$J_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\hbar|j,m-1\rangle$$

to evaluate the "lower" states, noting that

$$J_- = J_{1-} + J_{2-}$$

and that J_{i-} acts only on the m_i states.

$j = 3/2$ States

We also note that

$$j(j+1) - m(m-1) = j^2 + j - m^2 + m = (j+m)(j-m+1)$$

So we have a more convenient expression

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Thus

$$J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle' = (J_{1-} + J_{2-})\left|1,\tfrac{1}{2}\right\rangle$$

yields

$$\sqrt{3}\hbar\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle' = \sqrt{2}\hbar\left|0,\tfrac{1}{2}\right\rangle + \sqrt{1}\hbar\left|1,-\tfrac{1}{2}\right\rangle$$

which gives

$$\boxed{\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle' = \sqrt{\tfrac{2}{3}}\left|0,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,-\tfrac{1}{2}\right\rangle}$$

$j = 3/2$ States

Using the lowering operator again,

$$J_-\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle' = (J_{1-} + J_{2-})\left[\sqrt{\tfrac{2}{3}}\left|0,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,-\tfrac{1}{2}\right\rangle\right]$$

we have

$$\sqrt{4}\hbar\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{2}\hbar\sqrt{\tfrac{2}{3}}\left|-1,\tfrac{1}{2}\right\rangle + \sqrt{1}\hbar\sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle + \sqrt{2}\hbar\sqrt{\tfrac{1}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

which gives

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{\tfrac{1}{3}}\left|-1,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

$j = 3/2$ States

And once more,

$$J_-\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle' = (J_{1-}+J_{2-})\left[\sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|-1,\tfrac{1}{2}\right\rangle\right]$$

we have

$$\sqrt{3}\hbar\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle' = \left[\sqrt{2}\hbar\sqrt{\tfrac{2}{3}}\left|-1,-\tfrac{1}{2}\right\rangle + \sqrt{1}\hbar\sqrt{\tfrac{1}{3}}\left|-1,-\tfrac{1}{2}\right\rangle\right]$$

which gives

$$\boxed{\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle' = \left|-1,-\tfrac{1}{2}\right\rangle}$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Clebsch-Gordan Coefficients

We can pick out the Clebsch-Gordan coefficients from the relations between the $|jm\rangle$ and $|m_1 m_2\rangle$. For $j = 3/2$, we have

$$\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle' = \left|1,\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle' = \sqrt{\tfrac{2}{3}}\left|0,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|1,-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{\tfrac{1}{3}}\left|-1,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

$$\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle' = \left|-1,-\tfrac{1}{2}\right\rangle$$

We tabulate this on the right

$j = \frac{3}{2}$		m			
m_1	m_2	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
1	$\frac{1}{2}$	1			
1	$-\frac{1}{2}$		$\sqrt{\frac{1}{3}}$		
0	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$		
0	$-\frac{1}{2}$			$\sqrt{\frac{2}{3}}$	
-1	$\frac{1}{2}$			$\sqrt{\frac{1}{3}}$	
-1	$-\frac{1}{2}$				1

To declutter, we leave all non-essential zeroes as blanks

$j = 1/2$ States

We are now left with $j = 1/2$ states. However, we cannot use ladder operators to go from one j state to a state with a different value of j.

We do know that eigenstates of Hermitian operators are orthogonal

$$\langle j'm'|jm\rangle = \delta_{jj'}\delta_{mm'}$$

To move from a $j = 3/2$ state to a $j = 1/2$ state, let us then consider the orthogonality

$$\left\langle \tfrac{3}{2}, m \middle| \tfrac{1}{2}, m \right\rangle = 0$$

$j = 1/2$ States

Let us take the "highest" $m -$ state for $j = 1/2$. Since $m = m_1 + m_2$, we may write

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = a\left|0,\tfrac{1}{2}\right\rangle + b\left|1,-\tfrac{1}{2}\right\rangle$$

Then

$$\left\langle\tfrac{3}{2},\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = \left[\sqrt{\tfrac{2}{3}}\left\langle 0,\tfrac{1}{2}\right| + \sqrt{\tfrac{1}{3}}\left\langle 1,-\tfrac{1}{2}\right|\right]\left[a\left|0,\tfrac{1}{2}\right\rangle + b\left|1,-\tfrac{1}{2}\right\rangle\right] = \sqrt{\tfrac{2}{3}}\,a + \sqrt{\tfrac{1}{3}}\,b = 0$$

indicating that

$$b = -\sqrt{2}\,a$$

Thus,

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = a\left|0,\tfrac{1}{2}\right\rangle - \sqrt{2}\,a\left|1,-\tfrac{1}{2}\right\rangle$$

Normalizing, we have

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = \sqrt{\tfrac{1}{3}}\left|0,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,-\tfrac{1}{2}\right\rangle$$

$j = 1/2$ States

Using the lowering operator,

$$J_-\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = (J_{1-} + J_{2-})\left[\sqrt{\tfrac{1}{3}}\left|0,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,-\tfrac{1}{2}\right\rangle\right]$$

we have

$$\sqrt{1}\hbar\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{2}\hbar\sqrt{\tfrac{1}{3}}\left|-1,\tfrac{1}{2}\right\rangle + \sqrt{1}\hbar\sqrt{\tfrac{1}{3}}\left|0,-\tfrac{1}{2}\right\rangle - \sqrt{2}\hbar\sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

which gives

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{\tfrac{2}{3}}\left|-1,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle$$

Clebsch-Gordan Coefficients

For $j = 1/2$, we have

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle' = \sqrt{\tfrac{1}{3}}\left|0,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|1,-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle' = \left[\sqrt{\tfrac{2}{3}}\left|-1,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|0,-\tfrac{1}{2}\right\rangle\right]$$

We tabulate the Clebsch-Gordan coefficients on the right.

The complete CG table is shown on the next slide.

$J = \frac{1}{2}$		m	
m_1	m_2	$\frac{1}{2}$	$-\frac{1}{2}$
1	$\frac{1}{2}$		
1	$-\frac{1}{2}$	$-\sqrt{\frac{2}{3}}$	
0	$\frac{1}{2}$	$\sqrt{\frac{1}{3}}$	
0	$-\frac{1}{2}$		$-\sqrt{\frac{1}{3}}$
-1	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$
-1	$-\frac{1}{2}$		

To declutter, we leave all non-essential zeroes as blanks

Clebsch-Gordan Table

$j_1 = 1$ $j_2 = \frac{1}{2}$		$J = \frac{3}{2}$				$J = \frac{1}{2}$	
		m					
m_1	m_2	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
1	$\frac{1}{2}$	1					
1	$-\frac{1}{2}$		$\sqrt{\frac{1}{3}}$			$-\sqrt{\frac{2}{3}}$	
0	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$			$\sqrt{\frac{1}{3}}$	
0	$-\frac{1}{2}$			$\sqrt{\frac{2}{3}}$			$-\sqrt{\frac{1}{3}}$
-1	$\frac{1}{2}$			$\sqrt{\frac{1}{3}}$			$\sqrt{\frac{2}{3}}$
-1	$-\frac{1}{2}$				1		