Quantum Mechanics 2

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Addition of Angular Momenta

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Adding Angular Momenta

With the introduction of spin, one could surmise that the total angular momentum of an electron would be

$$J = L + S$$

When we consider multi-electron systems, one might also consider adding the orbital angular momenta of the electrons, or perhaps add the spins of the electrons.

If we consider the usual way of adding vectors component by component, this cannot be easily done in quantum mechanics because the components of angular momentum do not commute, and cannot be known at the same time.

In order to develop a quantum theory of adding angular momenta, one has start from the basic concept that the sum of two angular momenta must also be an angular momentum.

Total Angular Momentum

Thus, for

$$J = J_1 + J_2$$

The sum J must also obey the Lie Algebra of angular momenta

$$[J_+, J_-] = 2\hbar J_z$$

$$[J_z, J_\pm] = \pm\hbar J_\pm$$

$$[J^2, J_\pm] = 0$$

$$[J^2, J_z] = 0$$

Since orbital and spin angular momenta are independent of each other, and so do angular momenta of different objects, it is fairly reasonable to further assert that

$$[J_1, J_2] = 0$$

Quantum Numbers

We also assert that if j, m are the quantum numbers of the total angular momentum, then

$$j \geq 0$$

$$m = -j, -j+1, \cdots, j-1, j$$

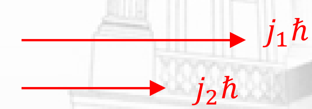
Now, because $[J^2, J_z] = 0$, it is actually possible to know the magnitude and one component of angular momentum at the same time. Conventionally, we take this to be the $z-$component.

Since the eigenvalue of J_z is $m\hbar$, then the sum of the $z-$component is

$$m\hbar = m_1\hbar + m_2\hbar$$

Now one could imagine that one of the scenarios in adding two vectors is the addition of two that have the same direction. If we take this direction as the $z-$axis, then each m_i would take on the maximum value j_i. It is also reasonable to expect that in this case, m would also take the maximum value j. Thus, one of the possible values of j is

$$j = j_1 + j_2$$



Counting States

For

$$j = j_1 + j_2$$

the number of $m -$ states is

$$2(j_1 + j_2) + 1$$

If one were to add two angular momenta, one would expect the corresponding eigenstates would be of a linear combination of

$$|j_1 m_1\rangle |j_2 m_2\rangle$$

and in terms of the quantum numbers of the total angular momentum, the states can be denoted by

$$|jm\rangle$$

There are a total of

$$(2j_1 + 1)(2j_2 + 1) = 4j_1 j_2 + 2j_1 + 2j_2 + 1$$

$m -$ states for the former, and

$$2j + 1$$

$m -$ states for the latter.

Comparing States

This indicates that

$$j = j_1 + j_2$$

Is not the only possible value of j. If it were so, then there would be $4j_1j_2$ unaccounted $m -$ states.

We now recall that j may only have either integral values or half-integral values. The rotational properties of these two sets are very different, as it would take two rotations for a half-integral state to return to the original state.

As such, these two sets are distinct from each other. So if j were to have different values, all values must be either all integers, or all half-integers. The different values of j must therefore differ by integers.

Taking this as a cue, let us consider the sum

$$\sum_{a}^{b}(2j+1)$$

taking into account the different orientations of the two vectors.

Comparing States

We require that

$$\sum_{a}^{b}(2j+1) = (2j_1+1)(2j_2+1)$$

Using

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}, \quad \sum_{n=1}^{N} 1 = N$$

We have

$$\sum_{a}^{b}(2j+1) = 2\left[\frac{b(b+1)}{2} - \frac{(a-1)a}{2}\right] + (b-a+1) = b^2 + 2b - a^2 + 1$$

Comparing States

With an end in mind of removing squared terms, we define

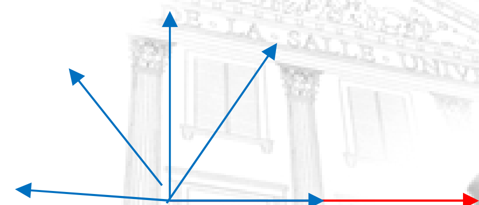
$$a = c - d$$
$$b = c + d$$

then

$$\sum_{a}^{b}(2j+1) = (c+d\,)^2 + 2(c+d\,) - (c-d)^2 + 1$$
$$= c^2 + 2cd + d^2 + 2c + 2d - c^2 + 2cd - d^2 + 1 = 4cd + 2c + 2d + 1$$
$$= (2c+1)(2d+1)$$

Thus, for $j_1 > j_2$, we have

$$c = \ j_1, \quad d = j_2$$



And this gives us, more generally

$$a = |j_1 - j_2\,|, \quad b = j_1 + j_2$$

The expression for a describes two vectors in opposite directions, while expression b describes two vectors in the same direction. All intermediate values of j correspond to two vectors oriented at an angle with each other.

Clebsch-Gordan Coefficients

To summarize, for $J = J_1 + J_2$, the possible values of the total angular momentum quantum number j are

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \cdots, j_1 + j_2 = 1, j_1 + j_2$$

There are two ways of describing two angular momentum states. One through the total angular momentum, the other through the individual angular momentum states. These two are but different representations, and can be related to each other via completeness relations of the eigenkets

$$|j_1 j_2 j m\rangle = \sum_{m_1}^{j_1} \sum_{m_2}^{j_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

The states $|j_1 j_2 j m\rangle$ may be referred to as coupled states, while $|j_1 j_2 m_1 m_2\rangle$ are uncoupled states. The inner products $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ are called the Clebsch-Gordan Coefficients.