Quantum Mechanics 2

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Pauli Spin Matrices

Properties

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Pauli Spin Matrices

The Pauli spin matrices are

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Together with the unit matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

They form a complete basis for the SU(2) group. This means that any 2×2 special and unitary complex matrix may be written as

$$A = A_0 1 + A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3$$

or

$$A = \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix}$$

Properties

Property 1: Product rule for non-identical spin matrices

$$\sigma_1\sigma_2 = -\sigma_1\sigma_2 = i\sigma_3 \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1 \quad \sigma_3\sigma_i = -\sigma_1\sigma_3 = i\sigma_2$$

or generally,

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k; \qquad i \neq j$$

Property 2: Squares

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$$

Property 3: General product rule

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$$

Properties

Property 4: Commutation

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

Property 5: Anticommutation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

Property 6: Dot product with a vector

$$\sigma \cdot a = \sum_{i=1}^{3} \sigma_i a_i = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$$

Then

$$(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$$

and

$$(\sigma \cdot a)^2 = a^2$$

Properties

Property 7: Hermitian

$$\sigma_i^\dagger = \sigma_i$$

Property 8: Special with negative signature

$$\det \sigma_i = -1$$

Property 9: Unitary

$$\sigma_i^\dagger \sigma_i = 1$$

Property 10: Traceless

$$Tr(\sigma_i) = 0$$