Quantum Mechanics 2

Robert C. Roleda
Physics Department

Spin

De La Salle University

Duplexity

By 1920, Physicists have come to a good understanding of the atom with the work of Niels Bohr. In 1922, Walter Gerlach conducted an experiment that was first conceptualized in 1921 by Otto Stern. Passing silver atoms through an inhomogeneous magnetic field, Gerlach observed an equal number of deflections going up and down. While this provides a crucial validation of the Bohr-Sommerfeld theory showing quantization, the number of states suggested by the results was double that of the theory.

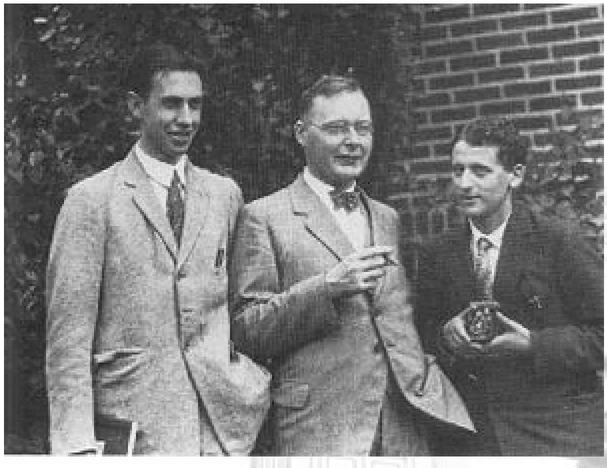
This mysterious doubling was known as *Mechanische Zweidentigkeit* in German, and *duplexity* in English.

A fourth quantum number

In 1924, Wolfgang Ernst Pauli formulated the Pauli Exclusion Principle that no two electrons can have identical quantum numbers.

Duplexity therefore necessitates the introduction of a fourth quantum number, with two possible values.

A year later, two doctoral students of Paul Ehrenfest in Leiden, George Uhlenbeck and Samuel Goudsmit hypothesized Pauli's new degree of freedom as the electron spin. In spite of strong objections of some prominent physicists, they managed to convince Bohr, and the rest is history.



Ralph Kronig who was doing his PhD at Columbia University actually came up with the idea several months before the Leiden duo but decided not to publish after being ridiculed by Pauli, saying "It is indeed very clever but of course has nothing to do with reality."

Spin

A spinning object carries angular momentum. In [angular momentum] , angular momenta are defined by their Lie Algebra which we now apply to spin

$$[S_+, S_-] = 2\hbar S_z$$

$$[S_z, S_\pm] = \pm\hbar S_\pm$$

$$[S^2, S_\pm] = 0$$

$$[S^2, S_z] = 0$$

where

$$S_\pm = S_x \pm iS_y$$

Quantum mechanical spin is different from classical spin because spinning is possible only with extended objects. Electrons however are considered as point particles. Moreover, classical spinning is a dynamical state, whereas electron spin is ever present and always the same value, regardless of its state. Thus while it is called spin, it suffices to assert that electron has an intrinsic angular momentum (meaning it is part of its existence in contrast to properties that are called dynamical – those relating to their state of motion) with a quantum number of 1/2.

Operator Equations

If we denote the simultaneous eigenkets of S^2, S_z as $|s, m\rangle$, then

$$S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$$

$$S_z|s,m\rangle = m\hbar|s,m\rangle$$

$$S_\pm|j,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\hbar|s,m\pm 1\rangle$$

For spin $s = \frac{1}{2}$,

$$S^2|\tfrac{1}{2},m\rangle = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2|\tfrac{1}{2},m\rangle = \tfrac{3}{4}\hbar^2|\tfrac{1}{2},m\rangle$$

$$S_z|\tfrac{1}{2},m\rangle = m\hbar|\tfrac{1}{2},m\rangle$$

$$S_\pm|\tfrac{1}{2},m\rangle = \sqrt{\tfrac{3}{4} - m(m\pm 1)}\hbar|\tfrac{1}{2},m\pm 1\rangle$$

The magnetic quantum number m may take on values of $\pm\frac{1}{2}$. It is convenient to introduce at this point an abbreviated notation. The value of s is fixed, so it does not need to be spelled out. The quantum number m on the other hand takes on two possible values, one positive, and the other is negative. We then write

$$|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle \Longrightarrow |\pm\rangle$$

Operator Equations

With this new notation,

$$S^2|\pm\rangle = \frac{3}{4}\hbar^2|\pm\rangle$$

$$S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

$$S_+|+\rangle = 0$$

$$S_+|-\rangle = \hbar|+\rangle$$

$$S_-|+\rangle = \hbar|-\rangle$$

$$S_-|-\rangle = 0$$

Matrix Representations

The matrix elements of each of these operators are

$$\langle +|S^2|+\rangle = \tfrac{3}{4}\hbar^2 \quad \langle +|S^2|-\rangle = 0 \quad \langle -|S^2|+\rangle = 0 \quad \langle -|S^2|-\rangle = \tfrac{3}{4}\hbar^2$$

$$\langle +|S_z|+\rangle = \frac{\hbar}{2} \quad \langle +|S_z|-\rangle = 0 \quad \langle -|S_z|+\rangle = 0 \quad \langle -|S_z|-\rangle = -\frac{\hbar}{2}$$

$$\langle +|S_+|+\rangle = 0 \quad \langle +|S_+|-\rangle = \hbar \quad \langle -|S_+|+\rangle = 0 \quad \langle -|S_+|-\rangle = 0$$

$$\langle +|S_-|+\rangle = 0 \quad \langle +|S_-|-\rangle = 0 \quad \langle -|S_-|+\rangle = \hbar \quad \langle -|S_-|-\rangle = 0$$

Thus,

$$S^2 = \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The eigenkets on the other hand are

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli Spin Matrices

The expressions

$$S_+ = S_x + iS_y \quad S_- = S_x - iS_y$$

may be inverted to give

$$S_x = \frac{S_+ + S_-}{2} \quad S_y = \frac{S_+ - S_-}{2i}$$

We then have

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The three rectangular components of the spin operator may thus be expressed as

$$S_i = \frac{\hbar}{2}\sigma_i$$

where σ_i are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$