Quantum Mechanics 2

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Associated Laguerre Polynomials

Properties

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Associated Laguerre Equation

The Associated Laguerre Equation

$$\rho\frac{d^2L}{d\rho^2} + (k+1-\rho)\frac{dL}{d\rho} + sL = 0$$

has the solutions

$$L_s^k(\rho) = \sum_{m=0}^{s}(-1)^m\frac{(s+k)!}{(s-m)!\,m!\,(k+m)!}\,\rho^m$$

The Associated Laguerre polynomials L_s^kmay also be evaluated from the Laguerre Polynomials through

$$L_s^k(\rho) = (-1)^k\frac{d^k}{d\rho^k}L_{s+k}(\rho)$$

The Laguerre polynomials L_n satisfies the equation

$$\rho\frac{d^2L}{d\rho^2} + (1-\rho)\frac{dL}{d\rho} + nL = 0$$

Recursion and Orthogonality

Alternatively, the Associated Laguerre polynomials can be evaluated using Rodrigues' formula

$$L_s^k(\rho) = \frac{1}{s!} e^{\rho} \rho^k \frac{d^s}{d\rho^s}\left(e^{-\rho}\rho^{s+k}\right)$$

A potent relation is the recursion relation

$$(s+1)L_{s+1}^k(\rho) = (2s+k+1-\rho)L_s^k(\rho) - (s+k)L_{s-1}^k(\rho)$$

and its derivative

$$\rho \frac{dL_s^k(\rho)}{d\rho} = sL_s^k(\rho) - (s+k)L_{s-1}^k(\rho)$$

Of particular importance is the orthogonality condition

$$\int_0^\infty e^{-\rho}\rho^k L_s^k(\rho) L_{s'}^k(\rho)\, d\rho = \frac{(s+k)!}{s!}\delta_{ss'}$$

Other Useful Relations

Another useful relations is

$$L_s^k(0) = \frac{(s+k)!}{s!\,k!}$$

The recursion relation can be rearranged as

$$\rho L_s^k(\rho) = (2s+k+1)L_s^k(\rho) - (s+k)L_{s-1}^k(\rho) - (s+1)L_{s+1}^k(\rho)$$

Thus,

$$\int_0^\infty e^{-\rho}\rho^{k+1}L_s^k(\rho)L_{s'}^k(\rho)d\rho = \int_0^\infty e^{-\rho}\rho^k\big[(2s+k+1)L_s^k(\rho) - (s+k)L_{s-1}^k(\rho) - (s+1)L_{s+1}^k(\rho)\big]L_{s'}^k(\rho)d\rho$$

Applying the orthogonality condition, we get have

$$\int_0^\infty e^{-\rho}\rho^{k+1}L_s^k(\rho)L_s^k(\rho)d\rho = \frac{(s+k)!}{s!}(2s+k+1)$$