Quantum Mechanics 2

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Spherical Harmonics

Part 3 (Properties)

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Spherical Harmonics

The eigenfunctions of $L^2$and L_z, and the solutions to the angular equation for central potentials are the spherical harmonics

$$Y_{l,m} = (-1)^{l+m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l$$

where l is a positive definite integer, and

$$m = -l, -l+1, \cdots, l-1, l$$

This module covers some special properties of these functions.

Complex Conjugate

Spherical harmonics are eigenfunctions of the L_z operator

$$L_z Y_{lm} = -i\hbar \frac{dY_{lm}}{d\varphi} = m\hbar Y_{lm}$$

Taking the complex conjugate,

$$i\hbar \frac{dY_{lm}^*}{d\varphi} = m\hbar Y_{lm}^*$$

This implies that

$$L_z Y_{lm}^* = -i\hbar \frac{dY_{lm}^*}{d\varphi} = -m\hbar Y_{lm}^*$$

suggesting that Y_{lm}^* are eigenfunctions of L_z with eigenvalues $-m\hbar$.

Complex Conjugate

The spherical harmonics are also eigenfunctions of L^2

$$L^2 Y_{l,m} = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{lm} = l(l+1)\hbar^2 Y_{l,m}$$

Taking the complex conjugate,

$$-\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{lm}^* = l(l+1)\hbar^2 Y_{lm}^*$$

Thus,

$$L^2 Y_{lm}^* = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{lm}^* = l(l+1)\hbar^2 Y_{lm}^*$$

suggesting that Y_{lm}^* are eigenfunctions of L^2 with eigenvalues $l(l+1)\hbar^2$. Thus, in all

$$Y_{lm}^*(\theta,\varphi) = C_m Y_{l,-m}(\theta,\varphi)$$

Complex Conjugate

Now,

$$L_+Y_{l,m} = -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]Y_{l,m} = \sqrt{(l-m)(l+m+1)}\hbar Y_{l,m+1}$$

Complex conjugation gives

$$i\hbar e^{-i\varphi}\left[-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]Y_{l,m}^* = \sqrt{(l-m)(l+m+1)}\hbar Y_{l,m+1}^*$$

On the other hand,

$$i\hbar e^{-i\varphi}\left[-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]Y_{l,m}^* = i\hbar e^{-i\varphi}\left[-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]C_m Y_{l,-m}$$

$$= -C_m L_- Y_{l,-m} = -C_m\sqrt{(l-m)(l+m+1)}\hbar Y_{l,-m-1}$$

Thus,

$$Y_{l,m+1}^* = -C_m Y_{l,-m-1}$$

Complex Conjugate

Aside from

$$Y^*_{l,m+1} = -C_m Y_{l,-m-1}$$

We also have

$$Y^*_{lm}(\theta,\varphi) = C_m Y_{l,-m}(\theta,\varphi)$$

which gives

$$Y^*_{l,m+1}(\theta,\varphi) = C_{m+1} Y_{l,-(m+1)}(\theta,\varphi)$$

Comparison between the first and the third yields the recursion relation

$$C_{m+1} = -C_m$$

For $m = 0$, $e^{im\varphi} = 1$, so

$$Y^*_{l,m}(\theta,\varphi) = P_{l,0}(\theta) = C_0\, Y_{l,0}(\theta,\varphi) = C_0 P_{l,0}(\theta)$$

Hence, $C_0 = 1$. Using the recursion relation, we have

$$C_1 = -C_0 = -1; \quad C_2 = -C_1 = 1; \quad C_3 = -C_2 = -1; \text{ and so on } \cdots$$

Conjugate, Orthogonality, Completeness

In general, spherical harmonics satisfy the following complex conjugation relation

$$Y_{l,,m}^{*}(\theta,\varphi) = (-1)^{m} Y_{l,-m}(\theta,\varphi)$$

The spherical harmonics are orthogonal

$$\int_{0}^{2\pi} d\varphi \int_{0}^{\pi} Y_{l,m}^{*}(\theta,\varphi) Y_{l',m'}(\theta,\varphi) \sin\theta \; d\theta = \delta_{ll'}\delta_{mm'}$$

and the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m--l}^{l} Y_{l,m}^{*}(\theta',\varphi') Y_{l,m}(\theta,\varphi) = \delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi')$$