

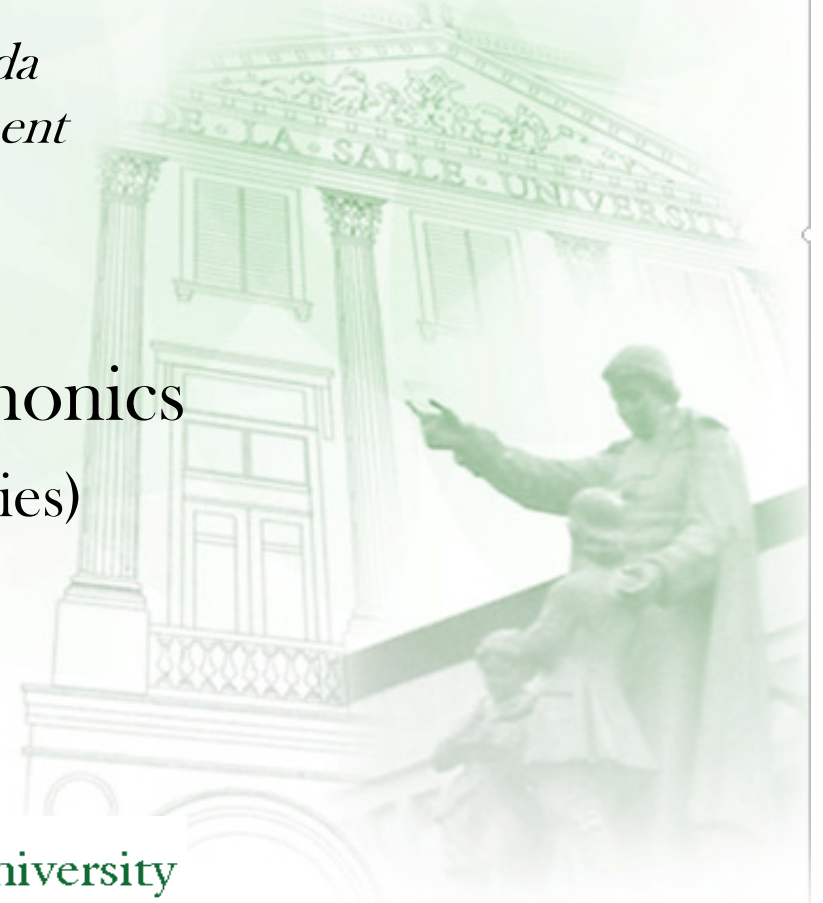
Quantum Mechanics 2

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Spherical Harmonics Part 3 (Properties)



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Spherical Harmonics

The eigenfunctions of L^2 and L_z , and the solutions to the angular equation for central potentials are the spherical harmonics

$$Y_{l,m} = (-1)^{l+m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l$$

where l is a positive definite integer, and

$$m = -l, -l+1, \dots, l-1, l$$

This module covers some special properties of these functions.



Complex Conjugate

Spherical harmonics are eigenfunctions of the L_z operator

$$L_z Y_{lm} = -i\hbar \frac{dY_{lm}}{d\varphi} = m\hbar Y_{lm}$$

Taking the complex conjugate,

$$i\hbar \frac{dY_{lm}^*}{d\varphi} = m\hbar Y_{lm}^*$$

This implies that

$$L_z Y_{lm}^* = -i\hbar \frac{dY_{lm}^*}{d\varphi} = -m\hbar Y_{lm}^*$$

suggesting that Y_{lm}^* are eigenfunctions of L_z with eigenvalues $-m\hbar$.



Complex Conjugate

The spherical harmonics are also eigenfunctions of L^2

$$L^2 Y_{l,m} = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{l,m} = l(l+1)\hbar^2 Y_{l,m}$$

Taking the complex conjugate,

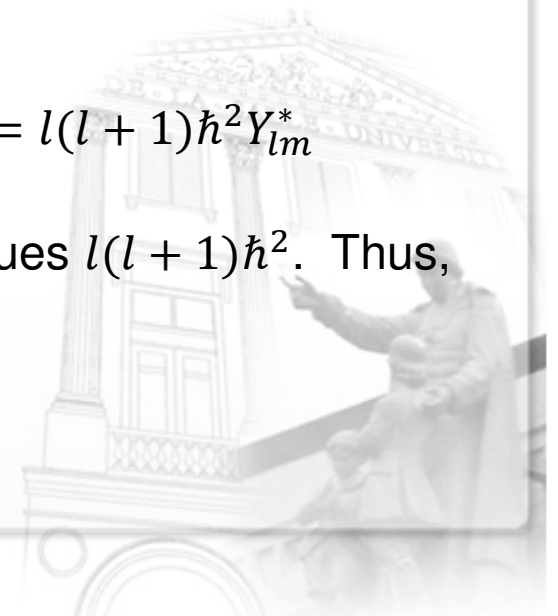
$$-\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{l,m}^* = l(l+1)\hbar^2 Y_{l,m}^*$$

Thus,

$$L^2 Y_{l,m}^* = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{l,m}^* = l(l+1)\hbar^2 Y_{l,m}^*$$

suggesting that $Y_{l,m}^*$ are eigenfunctions of L^2 with eigenvalues $l(l+1)\hbar^2$. Thus, in all

$$Y_{l,m}^*(\theta, \varphi) = C_m Y_{l,-m}(\theta, \varphi)$$



Complex Conjugate

Now,

$$L_+ Y_{l,m} = -i\hbar e^{i\varphi} \left[i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] Y_{l,m} = \sqrt{(l-m)(l+m+1)} \hbar Y_{l,m+1}$$

Complex conjugation gives

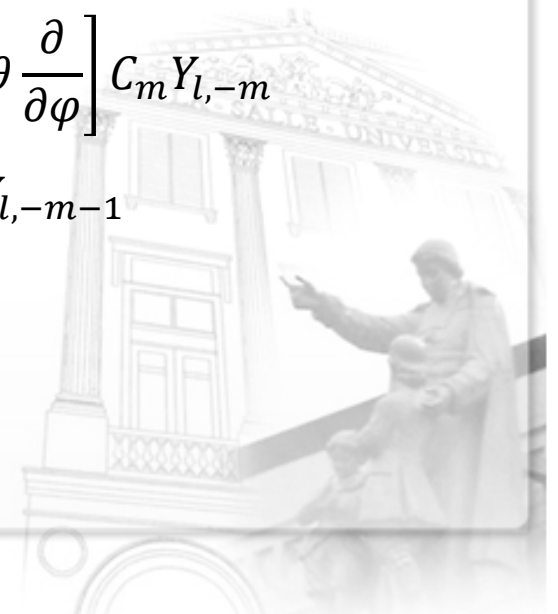
$$i\hbar e^{-i\varphi} \left[-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] Y_{l,m}^* = \sqrt{(l-m)(l+m+1)} \hbar Y_{l,m+1}^*$$

On the other hand,

$$\begin{aligned} i\hbar e^{-i\varphi} \left[-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] Y_{l,m}^* &= i\hbar e^{-i\varphi} \left[-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] C_m Y_{l,-m} \\ &= -C_m L_- Y_{l,-m} = -C_m \sqrt{(l-m)(l+m+1)} \hbar Y_{l,-m-1} \end{aligned}$$

Thus,

$$Y_{l,m+1}^* = -C_m Y_{l,-m-1}$$



Complex Conjugate

Aside from

$$Y_{l,m+1}^* = -C_m Y_{l,-m-1}$$

We also have

$$Y_{lm}^*(\theta, \varphi) = C_m Y_{l,-m}(\theta, \varphi)$$

which gives

$$Y_{l,m+1}^*(\theta, \varphi) = C_{m+1} Y_{l,-(m+1)}(\theta, \varphi)$$

Comparison between the first and the third yields the recursion relation

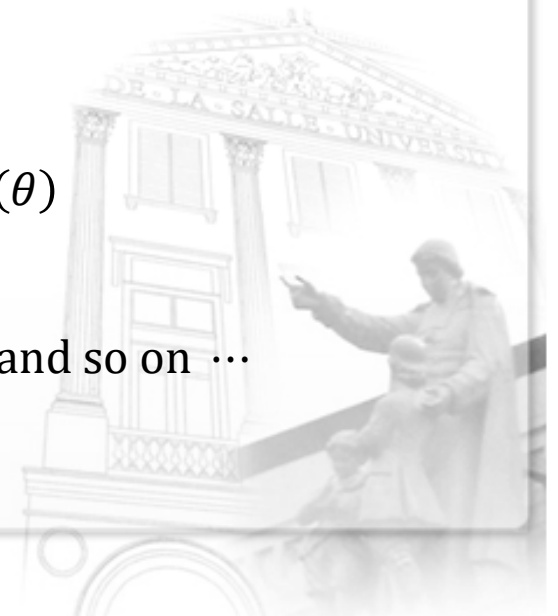
$$C_{m+1} = -C_m$$

For $m = 0$, $e^{im\varphi} = 1$, so

$$Y_{l,m}^*(\theta, \varphi) = P_{l,0}(\theta) = C_0 Y_{l,0}(\theta, \varphi) = C_0 P_{l,0}(\theta)$$

Hence, $C_0 = 1$. Using the recursion relation, we have

$$C_1 = -C_0 = -1; \quad C_2 = -C_1 = 1; \quad C_3 = -C_2 = -1; \quad \text{and so on } \dots$$



Conjugate, Orthogonality, Completeness

In general, spherical harmonics satisfy the following complex conjugation relation

$$Y_{l,m}^*(\theta, \varphi) = (-1)^m Y_{l,-m}(\theta, \varphi)$$

The spherical harmonics are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^\pi Y_{l,m}^*(\theta, \varphi) Y_{l',m'}(\theta, \varphi) \sin \theta d\theta = \delta_{ll'} \delta_{mm'}$$

and the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{l,m}^*(\theta', \varphi') Y_{l,m}(\theta, \varphi) = \delta(\cos \theta - \cos \theta') \delta(\varphi - \varphi')$$

