Quantum Mechanics 2

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Spherical Harmonics

Part 2 (All States)

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Raising Operation

To find the other states, we use the raising operator, which act on the L^2 and L_z eigenkets in the following way

$$L_+ \Big|l,m\rangle = \sqrt{l(l+1)-m(m+1)}\hbar\Big|\, l,m+1\rangle$$

Before proceeding, we first express the radicand to a product form

$$l(l+1)-m(m+1) = l^2+l-m^2-m = (l^2-m^2)+(l-m)$$
$$= (l+m)(l-m)+(l-m) = (l-m)(l+m+1)$$

Thus,

$$L_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l,m+1\rangle$$

Raising Operation

$$L_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l,m+1\rangle$$

and

$$L_+Y_{l,-l}(\theta,\varphi) = \sqrt{2l}\hbar Y_{l,-l+1}(\theta,\varphi)$$

Now,

$$L_+Y_{l,-l}(\theta,\varphi) = -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]C_l\sin^l\theta \; e^{-il\varphi}$$

$$= \hbar C_l e^{i(-l+1)\varphi}\left[\frac{d}{d\theta} + l\cot\theta\right]\sin^l\theta$$

Thus,

$$\sqrt{2l}\hbar Y_{l,-l+1}(\theta,\varphi) = \hbar C_l e^{i(-l+1)\varphi}\left[\frac{d}{d\theta} + l\cot\theta\right]\sin^l\theta$$

Second Lowest m –State

We next note that

$$l \cot\theta\,(\sin\theta)^k = k\cos\theta\,(\sin\theta)^{k-1} = \frac{d}{d\theta}(\sin\theta)^k$$

and

$$(\sin\theta)^k\left[\frac{d}{d\theta} + k\cot\theta\right]f(\theta) = (\sin\theta)^k\frac{df}{d\theta} + f\frac{d}{d\theta}(\sin\theta)^k = \frac{d}{d\theta}[f(\theta)(\sin\theta)^k]$$

The second lowest m – state is then

$$Y_{l,-l+1}(\theta,\varphi) = \frac{C_l}{\sqrt{2l}}e^{i(-l+1)\varphi}\frac{1}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^{2l}$$

Raising Operation no. 2

Raising a second time

$$L_+Y_{l,-l+1}(\theta,\varphi) = \sqrt{(l+l-1)(l-l+1+1)}\hbar Y_{l,-l+2}(\theta,\varphi)$$
$$= \sqrt{2(2l-1)}\hbar Y_{l,-l+2}(\theta,\varphi)$$

and

$$L_+Y_{l,-l+1}(\theta,\varphi) = -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]\left[\frac{C_l}{\sqrt{2l}}e^{i(-l+1)\varphi}\frac{1}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^{2l}\right]$$
$$= \hbar\frac{C_l}{\sqrt{2l}}e^{i(-l+2)\varphi}\left[\frac{d}{d\theta} + (l-1)\cot\theta\right]\left[\frac{1}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^{2l}\right]$$
$$= \hbar\frac{C_l}{\sqrt{2l}}e^{i(-l+2)\varphi}\frac{1}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{(\sin\theta)^{l-1}}{(\sin\theta)^l}\frac{d}{d\theta}(\sin\theta)^{2l}\right]$$

Thus,

$$Y_{l,-l+2}(\theta,\varphi) = \frac{C_l}{\sqrt{2(2l)(2l-1)}}e^{i(-l+2)\varphi}\frac{1}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2l}\right]$$

Raising Operation no. 3

Raising one more time

$$\begin{aligned} L_+Y_{l,-l+2}(\theta,\varphi) &= \sqrt{(l+l-2)(l-l+2+1)}\,\hbar Y_{l,-l+3}(\theta,\varphi) \\ &= \sqrt{3(2l-2)}\,\hbar Y_{l,-l+3}(\theta,\varphi) \end{aligned}$$

$$\begin{aligned} &L_+Y_{l,-l+2}(\theta,\varphi) \\ &= -i\hbar e^{i\varphi}\left[i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right]\left[\frac{C_l}{\sqrt{2(2l)(2l-1)}}e^{i(-l+2)\varphi}\frac{1}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2l}\right]\right] \\ &= \hbar\frac{C_l}{\sqrt{2(2l)(2l-1)}}e^{i(-l+3)\varphi}\left[\frac{d}{d\theta} + (l-2)\cot\theta\right]\left[\frac{1}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2l}\right]\right] \\ &\quad = \hbar\frac{C_l}{\sqrt{2(2l)(2l-1)}}e^{i(-l+3)\varphi}\frac{1}{(\sin\theta)^{l-2}}\frac{d}{d\theta}\left[\frac{(\sin\theta)^{l-2}}{(\sin\theta)^{l-1}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2l}\right]\right] \end{aligned}$$

Thus,

$$\begin{aligned} &Y_{l,-l+3}(\theta,\varphi) \\ &= \frac{C_l}{\sqrt{3!\,(2l)(2l-1)(2l-2)}}e^{i(-l+3)\varphi}\frac{1}{(\sin\theta)^{l-2}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2l}\right]\right] \end{aligned}$$

Change of Variable

The expression for the spherical harmonics would be simpler if we let

$$u = \cos\theta$$

The derivatives become

$$\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\cos\theta} = -\frac{d}{du}$$

Induction

The spherical harmonics are then

$$Y_{l,-l} = (-1)^0 C_l \; e^{-il\varphi} \frac{1}{(1-u^2)^{l/2}} \frac{d^0}{du^0}(1-u^2)^l$$

$$Y_{l,-l+1} = (-1)^1 \frac{C_l}{\sqrt{2l}} \; e^{i(-l+1)\varphi} \frac{1}{(1-u^2)^{(l-1)/2}} \frac{d}{du}(1-u^2)^l$$

$$Y_{l,-l+2} = (-1)^2 \frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+2)\varphi} \frac{1}{(1-u^2)^{(l-2)/2}} \frac{d^2}{du^2}(1-u^2)^l$$

$$Y_{l,-l+3} = (-1)^3 \frac{C_l}{\sqrt{3!\,(2l)(2l-1)(2l-2)}} e^{i(-l+3)\varphi} \frac{1}{(1-u^2)^{(l-3)/2}} \frac{d^3}{du^3}(1-u^2)^l$$

$\vdots$

$$Y_{l,m} = (-1)^{l+m} \frac{C_l}{\sqrt{(l+m)!}} \sqrt{\frac{(l-m)!}{(2l)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}}(1-u^2)^l$$

Spherical Harmonics

The eigenfunctions of L^2 and L_z, and the solutions to the angular equation for central potentials are therefore

$$Y_{l,m} = (-1)^{l+m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l$$

or

$$Y_{l,m} = (-1)^{m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (u^2-1)^l$$

The last expression may be compared to the Rodrigues' Formula for Associated Legendre polynomials

$$P_{l,m}(u) = \frac{1}{2^l l!} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (u^2-1)^l$$