

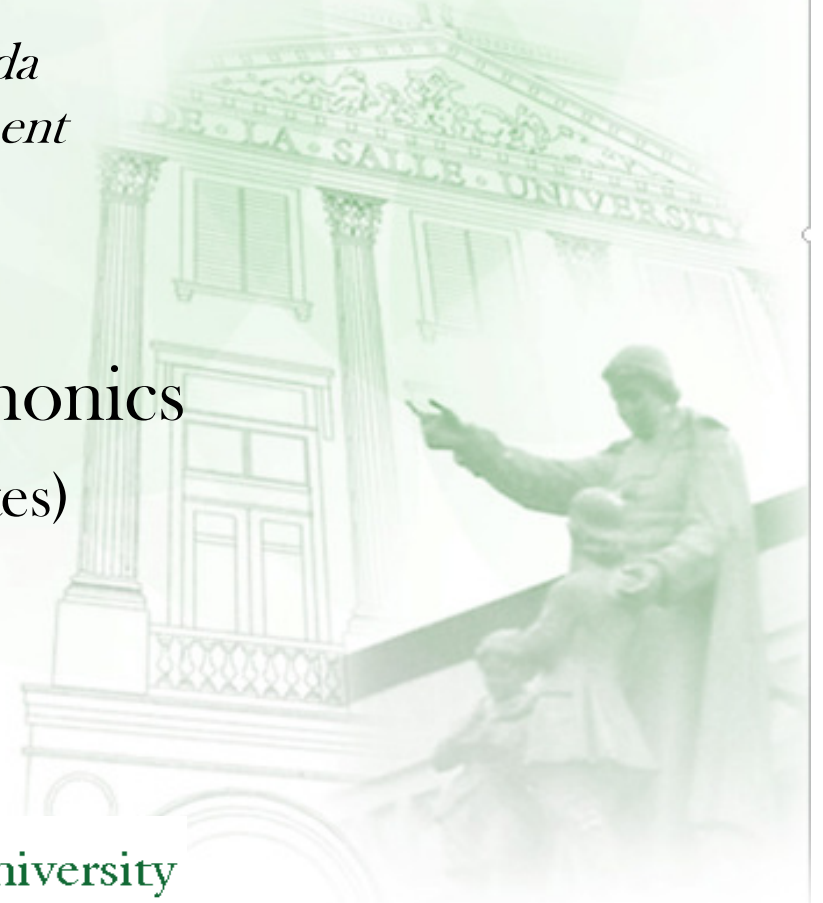
Quantum Mechanics 2

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Spherical Harmonics Part 2 (All States)



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Raising Operation

To find the other states, we use the raising operator, which act on the L^2 and L_z eigenkets in the following way

$$L_+ |l, m\rangle = \sqrt{l(l+1) - m(m+1)}\hbar |l, m+1\rangle$$

Before proceeding, we first express the radicand to a product form

$$\begin{aligned} l(l+1) - m(m+1) &= l^2 + l - m^2 - m = (l^2 - m^2) + (l - m) \\ &= (l+m)(l-m) + (l-m) = (l-m)(l+m+1) \end{aligned}$$

Thus,

$$L_+ |l, m\rangle = \sqrt{(l-m)(l+m+1)}\hbar |l, m+1\rangle$$



Raising Operation

$$L_+|l, m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l, m+1\rangle$$

and

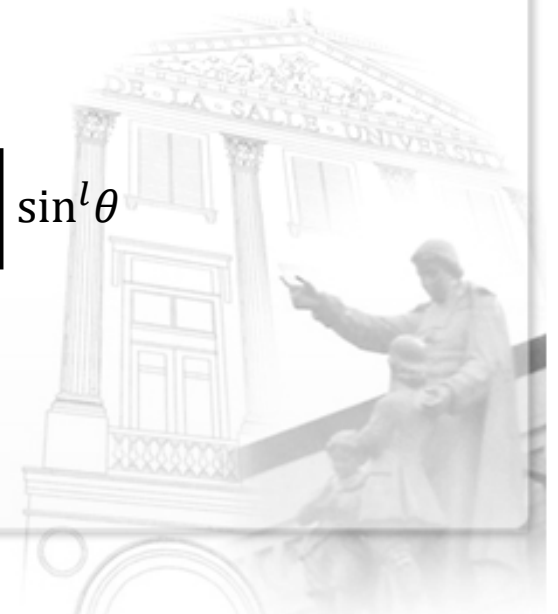
$$L_+Y_{l,-l}(\theta, \varphi) = \sqrt{2l}\hbar Y_{l,-l+1}(\theta, \varphi)$$

Now,

$$\begin{aligned} L_+Y_{l,-l}(\theta, \varphi) &= -i\hbar e^{i\varphi} \left[i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] C_l \sin^l \theta e^{-il\varphi} \\ &= \hbar C_l e^{i(-l+1)\varphi} \left[\frac{d}{d\theta} + l \cot \theta \right] \sin^l \theta \end{aligned}$$

Thus,

$$\sqrt{2l}\hbar Y_{l,-l+1}(\theta, \varphi) = \hbar C_l e^{i(-l+1)\varphi} \left[\frac{d}{d\theta} + l \cot \theta \right] \sin^l \theta$$



Second Lowest m – State

We next note that

$$l \cot \theta (\sin \theta)^k = k \cos \theta (\sin \theta)^{k-1} = \frac{d}{d\theta} (\sin \theta)^k$$

and

$$(\sin \theta)^k \left[\frac{d}{d\theta} + k \cot \theta \right] f(\theta) = (\sin \theta)^k \frac{df}{d\theta} + f \frac{d}{d\theta} (\sin \theta)^k = \frac{d}{d\theta} [f(\theta)(\sin \theta)^k]$$

The second lowest m – state is then

$$Y_{l,-l+1}(\theta, \varphi) = \frac{C_l}{\sqrt{2l}} e^{i(-l+1)\varphi} \frac{1}{(\sin \theta)^l} \frac{d}{d\theta} (\sin \theta)^{2l}$$



Raising Operation no. 2

Raising a second time

$$\begin{aligned} L_+ Y_{l,-l+1}(\theta, \varphi) &= \sqrt{(l+l-1)(l-l+1+1)} \hbar Y_{l,-l+2}(\theta, \varphi) \\ &= \sqrt{2(2l-1)} \hbar Y_{l,-l+2}(\theta, \varphi) \end{aligned}$$

and

$$\begin{aligned} L_+ Y_{l,-l+1}(\theta, \varphi) &= -i\hbar e^{i\varphi} \left[i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] \left[\frac{C_l}{\sqrt{2l}} e^{i(-l+1)\varphi} \frac{1}{(\sin \theta)^l} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \\ &= \hbar \frac{C_l}{\sqrt{2l}} e^{i(-l+2)\varphi} \left[\frac{d}{d\theta} + (l-1) \cot \theta \right] \left[\frac{1}{(\sin \theta)^l} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \\ &= \hbar \frac{C_l}{\sqrt{2l}} e^{i(-l+2)\varphi} \frac{1}{(\sin \theta)^{l-1}} \frac{d}{d\theta} \left[\frac{(\sin \theta)^{l-1}}{(\sin \theta)^l} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \end{aligned}$$

Thus,

$$Y_{l,-l+2}(\theta, \varphi) = \frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+2)\varphi} \frac{1}{(\sin \theta)^{l-1}} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta)^{2l} \right]$$



Raising Operation no. 3

Raising one more time

$$\begin{aligned} L_+ Y_{l,-l+2}(\theta, \varphi) &= \sqrt{(l+l-2)(l-l+2+1)} \hbar Y_{l,-l+3}(\theta, \varphi) \\ &= \sqrt{3(2l-2)} \hbar Y_{l,-l+3}(\theta, \varphi) \end{aligned}$$

$$\begin{aligned} &L_+ Y_{l,-l+2}(\theta, \varphi) \\ &= -i\hbar e^{i\varphi} \left[i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] \left[\frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+2)\varphi} \frac{1}{(\sin \theta)^{l-1}} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \right] \\ &= \hbar \frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+3)\varphi} \left[\frac{d}{d\theta} + (l-2) \cot \theta \right] \left[\frac{1}{(\sin \theta)^{l-1}} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \right] \\ &= \hbar \frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+3)\varphi} \frac{1}{(\sin \theta)^{l-2}} \frac{d}{d\theta} \left[\frac{(\sin \theta)^{l-2}}{(\sin \theta)^{l-1}} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \right] \end{aligned}$$

Thus,

$$\begin{aligned} &Y_{l,-l+3}(\theta, \varphi) \\ &= \frac{C_l}{\sqrt{3!(2l)(2l-1)(2l-2)}} e^{i(-l+3)\varphi} \frac{1}{(\sin \theta)^{l-2}} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta)^{2l} \right] \right] \end{aligned}$$



Change of Variable

The expression for the spherical harmonics would be simpler if we let

$$u = \cos \theta$$

The derivatives become

$$\frac{1}{\sin \theta} \frac{d}{d\theta} = -\frac{d}{d \cos \theta} = -\frac{d}{du}$$



Induction

The spherical harmonics are then

$$Y_{l,-l} = (-1)^0 C_l e^{-il\varphi} \frac{1}{(1-u^2)^{l/2}} \frac{d^0}{du^0} (1-u^2)^l$$

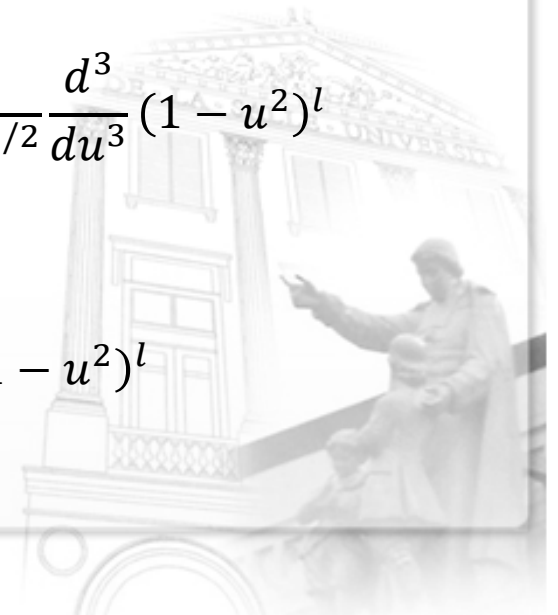
$$Y_{l,-l+1} = (-1)^1 \frac{C_l}{\sqrt{2l}} e^{i(-l+1)\varphi} \frac{1}{(1-u^2)^{(l-1)/2}} \frac{d}{du} (1-u^2)^l$$

$$Y_{l,-l+2} = (-1)^2 \frac{C_l}{\sqrt{2(2l)(2l-1)}} e^{i(-l+2)\varphi} \frac{1}{(1-u^2)^{(l-2)/2}} \frac{d^2}{du^2} (1-u^2)^l$$

$$Y_{l,-l+3} = (-1)^3 \frac{C_l}{\sqrt{3!(2l)(2l-1)(2l-2)}} e^{i(-l+3)\varphi} \frac{1}{(1-u^2)^{(l-3)/2}} \frac{d^3}{du^3} (1-u^2)^l$$

⋮

$$Y_{l,m} = (-1)^{l+m} \frac{C_l}{\sqrt{(l+m)!}} \sqrt{\frac{(l-m)!}{(2l)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l$$



Spherical Harmonics

The eigenfunctions of L^2 and L_z , and the solutions to the angular equation for central potentials are therefore

$$Y_{l,m} = (-1)^{l+m} \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2)^l$$

or

$$Y_{l,m} = (-1)^m \frac{1}{2^l l!} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\varphi} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (u^2-1)^l$$

The last expression may be compared to the Rodrigues' Formula for Associated Legendre polynomials

$$P_{l,m}(u) = \frac{1}{2^l l!} (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (u^2-1)^l$$

