

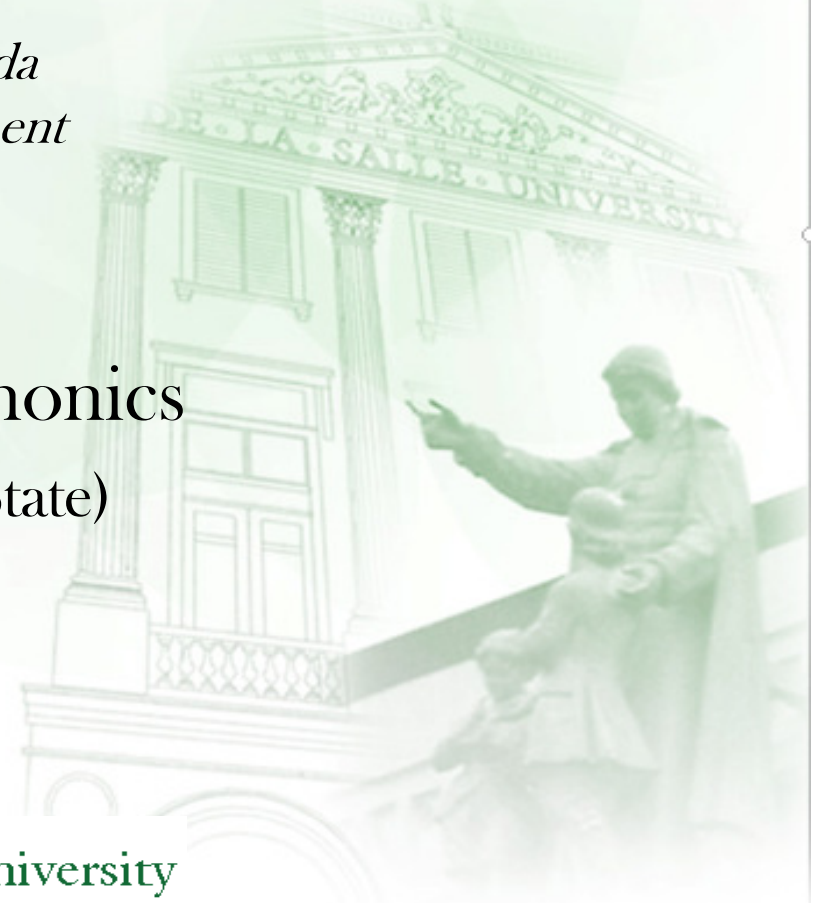
Quantum Mechanics 2

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Spherical Harmonics Part 1 (Lowest State)



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L_z States

In [angular momentum], we have evaluated the eigenvalues of the operators L^2 and L_z

$$L^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle$$

$$L_z|l, m\rangle = m\hbar|l, m\rangle$$

Let us now evaluate the eigenfunctions. For orbital angular momentum,

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

Thus, if we separate the eigenfunctions as

$$\langle \theta, \varphi | l, m \rangle = Y_{lm}(\theta, \varphi) = P_{lm}(\theta)\Phi(\varphi)$$

we have

$$-i\hbar \frac{d\Phi}{d\varphi} = m\hbar\Phi$$

which yields

$$\Phi(\varphi) = e^{im\varphi}$$



Angular Momentum Quantum Numbers

As we will see later in this course, the quantum number m manifests itself in the presence of magnetic fields. It is therefore called the **magnetic quantum number**. It was shown previously that it may take on values

$$m = -j, -j + 1, -j + 2, \dots, j - 1, j$$

In the case of orbital angular momentum, we write $j = l$. Now, the **angular momentum quantum number** j was shown to be either a positive definite half-integer, or a positive definite integer.

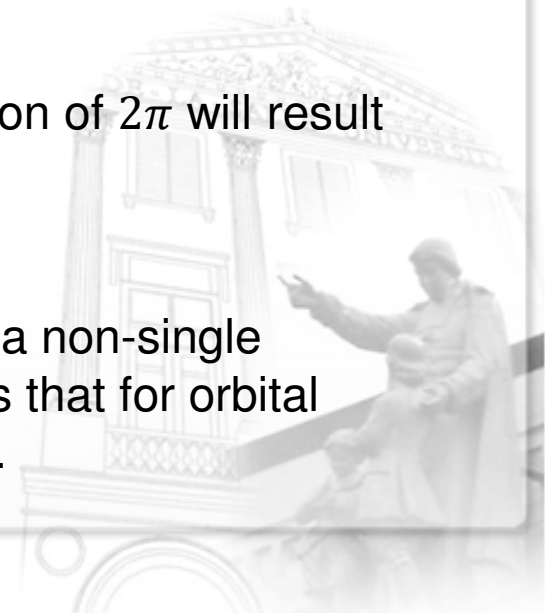
For orbital angular momentum, the φ – dependent part of eigenfunction is

$$\Phi(\varphi) = e^{im\varphi}$$

Orbital angular momenta are generators of rotation. A rotation of 2π will result in

$$\Phi(\varphi + 2\pi) = e^{im\varphi} e^{i2m\pi}$$

If l is a half-integer, so will m . Thus, $e^{i2m\pi} = -1$, leading to a non-single valued eigenfunction as $\Phi(\varphi + 2\pi) = -\Phi(\varphi)$. This suggests that for orbital angular momentum, **l must be a positive-definite integer**.



Ladder Operators

Now let us look for the θ – dependent part of the eigenfunctions. For this, we will need the ladder operators

$$J_{\pm} = J_x \pm iJ_y$$

which operates on the J^2 and J_z eigenkets as follows

$$J_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \hbar |j, m \pm 1\rangle$$

For orbital angular momentum,

$$L_{\pm} = -i\hbar e^{\pm i\varphi} \left[\pm i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right]$$



The Lowest m – State

We begin with the eigenket with the lowest value of m . Since it cannot go any lower,

$$L_-|l, -l\rangle = 0$$

This translates to

$$L_-Y_{l,-l}(\theta, \varphi) = -i\hbar e^{i\varphi} \left[-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \varphi} \right] P_{l,-l}(\theta) e^{-il\varphi} = 0$$

which gives us

$$\frac{dP_{l,-l}(\theta)}{d\theta} = l \cot \theta P_{l,-l}(\theta)$$

This separable ODE yields

$$\ln P_{l,-l} = l \ln |\sin \theta| + \ln C_l$$

and so

$$P_{l,-l}(\theta) = C_l \sin^l \theta$$



Normalization

We may evaluate C_l through normalization of the eigenfunction

$$\begin{aligned}\int Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega &= \int_0^\pi |P_{l,-l}(\theta)|^2 \sin \theta d\theta \int_0^{2\pi} |e^{-il\varphi}|^2 d\varphi \\ &= 2\pi \int_0^\pi |C_l|^2 \sin^{2l} \theta \sin \theta d\theta = 1\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^\pi (\sin \theta)^{2l} \sin \theta d\theta &= -(\sin \theta)^{2l} \cos \theta \Big|_0^\pi + 2l \int_0^\pi (\sin \theta)^{2l-1} (\cos \theta)^2 d\theta \\ &= 2l \left[\int_0^\pi (\sin \theta)^{2l-1} d\theta - \int_0^\pi (\sin \theta)^{2l+1} d\theta \right]\end{aligned}$$

Thus,

$$[2l + 1] \int_0^\pi (\sin \theta)^{2l+1} d\theta = 2l \int_0^\pi (\sin \theta)^{2l-1} d\theta$$



Normalization

If we apply

$$\int_0^\pi (\sin \theta)^{2l+1} d\theta = \frac{2l}{2l+1} \int_0^\pi (\sin \theta)^{2l-1} d\theta$$

successively,

$$\begin{aligned} \int_0^\pi (\sin \theta)^{2l+1} d\theta &= \frac{2l}{2l+1} \int_0^\pi (\sin \theta)^{2l-1} d\theta = \frac{(2l)(2l-2)}{(2l+1)(2l-1)} \int_0^\pi (\sin \theta)^{2l-3} d\theta \\ &= \dots = \frac{(2l)(2l-2) \dots 2}{(2l+1)(2l-1) \dots 3} \int_0^\pi \sin \theta d\theta = \frac{2l!! (2)}{(2l+1)!!} \end{aligned}$$

The double factorials may be recast as follows

$$\begin{aligned} \frac{2l!!}{(2l+1)!!} &= \frac{(2l)(2l-2) \dots 2}{(2l+1)(2l-1) \dots (3)(1)} = \frac{[(2l)(2l-2) \dots 2]^2}{(2l+1)(2l)(2l-1)(2l-2) \dots (3)(2)(1)} \\ &= \frac{[2^l(l)(l-1) \dots 1]^2}{(2l+1)!} = \frac{[2^l l!]^2}{(2l+1)!} \end{aligned}$$



Normalization

We thus have

$$1 = 2\pi \int_0^\pi |C_l|^2 \sin^{2l}\theta \sin\theta d\theta = |C_l|^2 4\pi \frac{[2^l l!]^2}{(2l+1)!}$$

and

$$C_l = [2^l l!]^{-1} \sqrt{\frac{(2l+1)!}{4\pi}}$$

We then have

$$Y_{l,-l} = [2^l l!]^{-1} \sqrt{\frac{(2l+1)!}{4\pi}} \sin^l\theta e^{-il\varphi}$$



Verification

We now verify that

$$Y_{l,-l} = (-1)^l [2^l l!]^{-1} \sqrt{\frac{(2l+1)!}{4\pi}} \sin^l \theta e^{-il\varphi}$$

is indeed an eigenfunction of L^2

$$\begin{aligned} L^2 Y_{l,-l} &= -\hbar^2 C_l \left[\frac{e^{-il\varphi}}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial (\sin \theta)^l}{\partial \theta} \right) + \frac{(\sin \theta)^l}{\sin^2 \theta} \frac{\partial^2 e^{-il\varphi}}{\partial \varphi^2} \right] \\ &= -\hbar^2 C_l \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (l(\sin \theta)^l \cos \theta) - l^2 (\sin \theta)^{l-2} \right] e^{-il\varphi} \\ &= -\hbar^2 C_l [l^2 (\sin \theta)^{l-2} (\cos \theta)^2 - l(\sin \theta)^l - l^2 (\sin \theta)^{l-2}] e^{-il\varphi} \\ &= \hbar^2 C_l [l^2 (\sin \theta)^{l-2} (\sin \theta)^2 + l(\sin \theta)^l] e^{-il\varphi} \\ &= l(l+1) \hbar^2 C_l (\sin \theta)^l e^{-il\varphi} = l(l+1) \hbar^2 Y_{l,-l} \end{aligned}$$

