Quantum Mechanics 2

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Angular Momentum

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Commutation Relations

Angular momentum is defined by

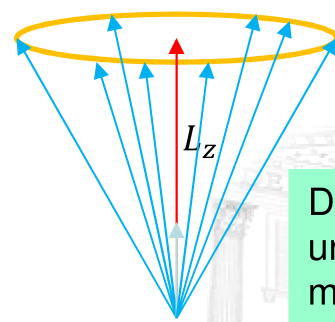
$$L = r \times p = \varepsilon_{ijk}\hat{e}_i x_j p_k$$

The position and momentum operators satisfy the following commutation rules

$$[x_i, p_j] = i\hbar\delta_{ij}; \quad [x_i, x_j] = 0; \quad [p_i, p_j] = 0$$

Thus, angular momentum operators obey

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$$



Direction is uncertain, but magnitude and one component are determinable

Also,

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

and

$$[L^2, L_i] = 0$$

While different components of angular momentum are not compatible, its magnitude square is compatible with all components.

New Operators

Let

$$L_{\pm} = L_x \pm iL_y$$

then the new set of angular momentum operators are L_+, L_-, L_z, L^2, and the full set of commutation relations between these four operators

$$[L_{+,} L_-] = 2\hbar L_z$$

$$[L_z, L_{\pm}] = \pm\hbar L_{\pm}$$

$$[L^2, L_{\pm}] = 0$$

$$[L^2, L_z] = 0$$

constitute the Lie Algebra of the angular momentum operators.

Eigenkets

Since L^2, L_z commute, they have common eigenkets, and we may label these using quantum numbers for each of the two operators. In particular, let

$$L^2|\lambda, m\rangle = \lambda\hbar^2|\lambda, m\rangle$$

$$L_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$$

From

$$[L^2, L_\pm] = 0$$

we have

$$[L^2, L_\pm]\big|\lambda, m\rangle = L^2 L_\pm\big|\lambda, m\rangle - L_\pm L^2|\lambda, m\rangle = 0$$

Thus,

$$L^2(L_\pm|\lambda, m\rangle) = L_\pm L^2\big|\lambda, m\rangle = L_\pm \lambda\hbar^2\big|\lambda, m\rangle = \lambda\hbar^2(L_\pm|\lambda, m\rangle)$$

indicating that $(L_\pm|\lambda, m\rangle)$ is an eigenket of L^2 with eigenvalue $\lambda\hbar^2$.

Eigenkets

From

$$[L_z, L_\pm] = \pm\hbar L_\pm$$

we have

$$[L_z, L_\pm]\big|\lambda, m\rangle = L_z L_\pm\big|\lambda, m\rangle - L_\pm L_z|\lambda, m\rangle = \pm\hbar L_\pm|\lambda, m\rangle$$

and

$$L_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$$

$$L_z(L_\pm|\lambda, m\rangle) = L_\pm L_z|\lambda, m\rangle \pm \hbar L_\pm|\lambda, m\rangle = (m \pm 1)\hbar(L_\pm|\lambda, m\rangle)$$

This suggests that $(L_\pm|\lambda, m\rangle)$ is an eigenket of L_z with an eigenvalue of $(m \pm 1)\hbar$, Thus,

$$L_\pm|\lambda, m\rangle = C_\pm(\lambda, m)\hbar|\lambda, m \pm 1\rangle$$

Eigenvalues

We now turn to the last commutation relation

$$[L_{+,}L_-] = 2\hbar L_z$$

But first, let us take a look at L_+L_-

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 + i(L_yL_x - L_xL_y)$$

The imaginary term can be eliminated if we also consider

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i(L_xL_y - L_yL_x)$$

Clearly,

$$L_+L_- + L_-L_+ = 2(L_x^2 + L_y^2)$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

which may be recast as

$$L_+L_- + L_-L_+ = 2(L^2 - L_z^2)$$

doing away with both L_x and L_y

Eigenvalues

$$L_+L_- + L_-L_+ = 2(L^2 - L_z^2)$$

Applying

$$[L_{+,}L_-] = L_+L_- - L_-L_+ = 2\hbar L_z$$

We find that

$$L_+L_- = L^2 - L_z^2 + \hbar L_z$$
$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

Sandwiching between bra and ket,

$$\langle\lambda m|L_+L_-|\lambda m\rangle = \langle\lambda m|(L^2 - L_z^2 + \hbar L_z)|\lambda m\rangle = (\lambda - m^2 + m)\hbar^2$$
$$\langle\lambda m|L_-L_+|\lambda m\rangle = \langle\lambda m|(L^2 - L_z^2 - \hbar L_z)|\lambda m\rangle = (\lambda - m^2 - m)\hbar^2$$

Eigenvalues

Now,

$$L_+^\dagger = \left(L_x + iL_y\right)^\dagger = L_x - iL_y = L_-$$

So

$$(L_-|\lambda m\rangle)^\dagger = \langle\lambda m|L_+$$

and if we write $L_-|\lambda m\rangle = |\psi\rangle$,

$$\langle\lambda m|L_+L_-|\lambda m\rangle = \langle\psi|\psi\rangle \geq 0$$

Likewise for $L_+|\lambda m\rangle = |\phi\rangle$,

$$\langle\lambda m|L_-L_+|\lambda m\rangle = \langle\phi|\phi\rangle \geq 0$$

$$\langle\lambda m|L_+L_-|\lambda m\rangle = (\lambda - m^2 + m)\hbar^2$$

We see then that

$$\langle\lambda m|L_+L_-|\lambda m\rangle = (\lambda - m^2 + m)\hbar^2 \geq 0$$

$$\langle\lambda m|L_-L_+|\lambda m\rangle = (\lambda - m^2 - m)\hbar^2 \geq 0$$

Eigenvalues

and

$$\lambda \geq m(m-1); \qquad \lambda \geq m(m+1)$$

This indicate that if $m > 0$, it must have an upper bound. If we consider $m < 0$, we may write $m = -|m|$, and $\lambda \geq m(m+1) = |m|(|m|-1)$. This suggests that $|m|$ has an upper bound, so that m must have a lower bound. Let the upper bound of m be b, and let the lower bound of m be a. This means that

$$L_+|\lambda b\rangle = 0; \qquad L_-|\lambda a\rangle = 0$$

$$L_\pm|\lambda, m\rangle = C_\pm(\lambda, m)\hbar|\lambda, m \pm 1\rangle$$

and

$$\lambda = a(a-1); \qquad \lambda = b(b+1)$$

This shows that

$$a = -b$$

and the values of m lie between these two extreme values

$$-b \leq m \leq b$$

Eigenvalues

To know more about b, we note that

$$L_{\pm}|\lambda, m\rangle = C_{\pm}(\lambda, m)\hbar|\lambda, m \pm 1\rangle$$

Successive application of L_+ allows us to go from $|\lambda a\rangle$ to $|\lambda b\rangle$ in incremental steps of one, and vice versa by using L_-. It is for this reason that $L_{\pm}$ are called **Ladder operators.** Knowing this, we write

$$b = a + n = -b + n$$

Thus,

$$b = \frac{n}{2}$$

where n is an integer. If we write replace b with j, we have

$$\lambda = j(j+1) \quad \longleftarrow \quad \lambda \geq m(m+1)$$

$$m = -j, -j+1, \cdots, j-1, j$$

where j is either a positive integer or a positive half-integer.

Generalized Angular Momentum

It must be noted that the results derived were based on the commutation relations of the angular momenta. Angular momenta defined by $L = r \times p$ do obey the set of commutation relations, but there could be other types of angular momenta which satisfy the same. This possibility may be inferred from the fact that there are two classes of angular momenta. Those with integral values of j, and those with half-integral values.

We can thus generalize the notion of angular momentum by asserting that the generalized angular momentum J is defined as those physical quantities whose operators obey the following set of commutation rules:

$$[J_+, J_-] = 2\hbar J_z$$

$$[J_z, J_\pm] = \pm\hbar J_\pm$$

$$[J^2, J_\pm] = 0$$

$$[J^2, J_z] = 0$$

where

$$J_\pm = J_x \pm iJ_y$$

Operator Equations

The operators J^2, J_z commute, and if we denote their simultaneous eigenkets as $|j,m\rangle$, then

$$J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$
$$J_z|j,m\rangle = m\hbar|j,m\rangle$$

$\lambda = j(j+1)$

As for the ladder operators,

$$J_\pm|j,m\rangle = C_\pm(j,m)\hbar|j,m\pm 1\rangle$$

To evaluate the coefficients $C_\pm(j,m)$, we note that

$$(J_-|jm\rangle)^\dagger = \langle jm|J_+$$

implies that

$$\langle jm|J_+J_-|jm\rangle = |C_-(j,m)|^2\hbar^2$$
$$\langle jm|J_-J_+|jm\rangle = |C_+(j,m)|^2\hbar^2$$

Ladder Operators

$$\lambda = j(j+1)$$

$$\langle \lambda m | L_+ L_- | \lambda m \rangle = (\lambda - m^2 + m)\hbar^2$$

At the same time,

$$\langle jm | J_+ J_- | jm \rangle = (j(j+1) - m^2 + m)\hbar^2$$
$$\langle jm | J_- J_+ | jm \rangle = (j(j+1) - m^2 - m)\hbar^2$$

It is clear then that

$$\langle jm | J_+ J_- | jm \rangle = |C_-(j,m)|^2 \hbar^2$$

$$C_-(j,m) = \sqrt{j(j+1) - m(m-1)}$$
$$C_+(j,m) = \sqrt{j(j+1) - m(m+1)}$$

and

$$J_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,\hbar\, |j, m\pm 1\rangle$$