Quantum Mechanics

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The Harmonic Oscillator Revisited

Part 6

Eigenstates

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Operator Method

We have until this point managed to do all calculations without needing to know how the eigenstates look like. In fact, in operator methods just like the one presented in this module, we don't need to know the expressions for eigenstates at all. All that we need to know is that they are eigenstates.

The advantage of operator methods is that we keep our focus on observables. Wave functions, which are not observables, are not essential at all in this type of approach. But if one insists of knowing the expressions for the eigenfunctions, the operator method is just as capable of generating these.

Much as we have done in this approach, we will continue to rely on the ladder operators.

Ground State

As before, we start with the ground state $|0\rangle$. We have seen in [ladder 3] that

$$A|0\rangle = 0$$

In wave-function representation,

$$Au_0(x) = \sqrt{\frac{m\omega}{2\hbar}}\, xu_0(x) + \frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx}u_0(x) = 0$$

This is a separable equation

$$\frac{du_0}{u_0} = -\frac{m\omega}{\hbar}x\, dx$$

and the solution is straightforward

$$u_0(x) = N \exp\left(-\frac{m\omega}{2\hbar}\, x^2\right)$$

Normalization

This ground state is a Gaussian function

$$u_0(x) = N \exp\left(-\frac{m\omega}{2\hbar}\, x^2\right)$$

The normalization constant can be easily obtained using the Gaussian Integral formula

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Hence,

$$N = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Excited States

The excited states can be evaluated using the raising operator

$$A^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i}{\sqrt{2m\omega\hbar}} p$$

Since

$$A^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

We have

$$A^\dagger |0\rangle = \sqrt{1}\,|1\rangle$$

$$A^\dagger |1\rangle = \sqrt{2}\,|2\rangle$$

$$A^\dagger |2\rangle = \sqrt{3}\,|3\rangle$$

and so on, then we have for example

$$|3\rangle = \frac{A^\dagger|2\rangle}{\sqrt{3}} = \frac{A^\dagger A^\dagger |1\rangle}{\sqrt{3\cdot 2}} = \frac{A^\dagger A^\dagger A^\dagger |0\rangle}{\sqrt{3\cdot 2\cdot 1}}$$

Excited States

In general,

$$|n\rangle = \frac{\left(A^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle$$

For example, the first-excited state can thus be generated from the ground state as follows:

$$u_1 = A^{\dagger}u_0 = \left[\sqrt{\frac{m\omega}{2\hbar}}\,x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right]u_0$$

$$u_1(x) = N\left[\sqrt{\frac{m\omega}{2\hbar}}\,x + \sqrt{\frac{\hbar}{2m\omega}}\frac{m\omega x}{\hbar}\right]\exp\left(-\frac{m\omega}{2\hbar}\,x^2\right)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{m\omega}{2\hbar}}\;2x\exp\left(-\frac{m\omega}{2\hbar}\,x^2\right)$$