Quantum Mechanics

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Fourier Transform

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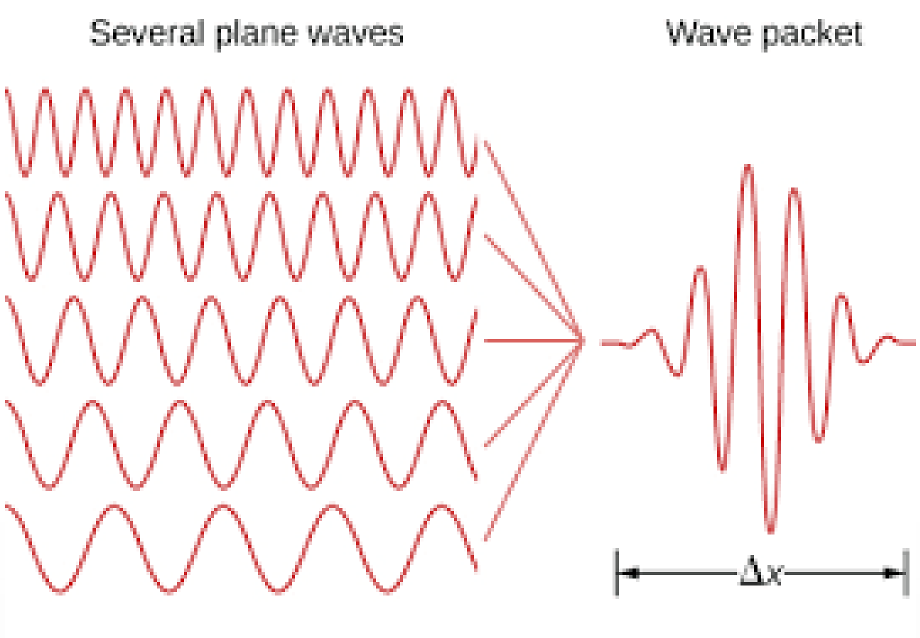
Fourier Theorem

Fourier theorem states that a periodic function $f(x)$ which is reasonably continuous may be expressed as a sum of a series of sine or cosine terms, each with specific amplitude and phase, known as the Fourier coefficients.



The sum is called the Fourier Series

$$f(x) = \sum_{i=1}^{N} a_n \sin(k_n x + \varphi_n)$$

Fourier Series

The amplitude-phase form

$$f(x) = \sum_{i=1}^{N} a_n \sin(k_n x + \varphi_n)$$

may be in a sine-cosine form

$$f(x) = \sum_{i=1}^{N} a_n \sin k_n x + \sum_{i=1}^{N} b_n \cos k_n x$$

where the amplitude and phase coefficients are replaced by two amplitude coefficients.

With Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, the Fourier series may also be cast in exponential form

$$f(x) = \sum_{i=N}^{N} c_n e^{ik_n x}$$

Fourier Transform

The Fourier series may be analytically continued to cases where wave vectors k are continuous, in which case the summation is replaced in an integral

$$f(x) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{ikx} dk$$

where $1/\sqrt{2\pi}$ is a constant chosen to keep the transform and its inverse similar. This is called the Fourier Transform

We now note that $\int e^{i(k-k')x} dk = 2\pi\delta(k-k')$

So

$$\int f(x) e^{-ik\prime x} dx = \frac{1}{\sqrt{2\pi}} \int dk g(k) \int e^{ikx}\, e^{-ik\prime x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int g(k) 2\pi\delta(k-k') dk = \sqrt{2\pi} g(k')$$

The inverse Fourier Transform is then

$$g(k) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} dx$$

Plane Wave

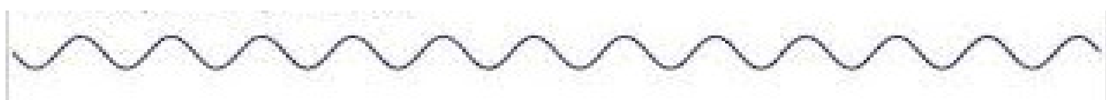
A plane wave is described in x – space by

$$\psi(x) = e^{i(qx-\omega t)}$$

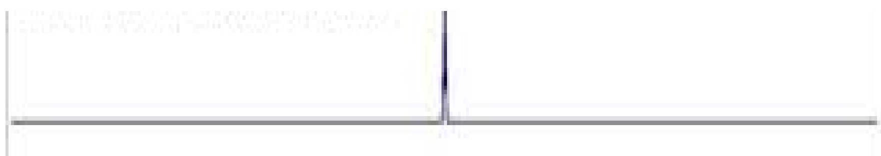
where the wave vector q and angular frequency ω are fixed. The corresponding k – space wave function is then

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int \psi(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int e^{iqx}e^{-ikx}dx\, e^{i\omega t} = \sqrt{2\pi}\delta(k-q)e^{i\omega t}$$

$\psi(x)$



$\phi(k)$



Thus in x – space, the system is infinitely extended $\delta x \to \infty$.

In k – space, the system is localized $\delta k = 0$.

Pulse

A travelling pulse wave is described in x – space by

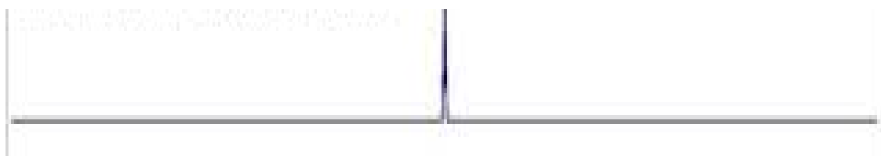
$$\psi(x) = \delta(x - vt)$$

The k – space wave function is then

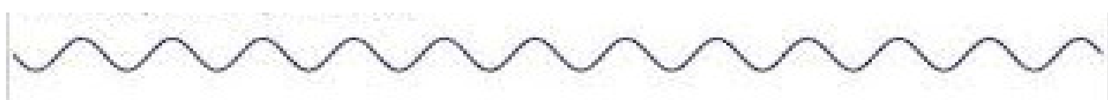
$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int \psi(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int \delta(x - vt)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}e^{-ikvt}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-i\omega t}$$

$\psi(x)$



$\phi(k)$



Thus in x – space, the system is localized $\delta x = 0$.

In k – space, the system is infinitely extended $\delta k \to \infty$.