Quantum Mechanics

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The Harmonic Oscillator Revisited
Part 4
Observables

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Ladder Operators

The operators introduced in [ladder1]

$$A = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i}{\sqrt{2m\omega\hbar}} p$$

$$A^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i}{\sqrt{2m\omega\hbar}} p$$

were found to be ladder operators of energy eigenkets in [ladder3], where

$$A|n\rangle = \sqrt{n}\, |n-1\rangle$$

$$A^{\dagger}|n\rangle = \sqrt{n+1}\, |n+1\rangle$$

Eigenenergies

The Hamiltonian was shown in [ladder1] to be

$$H = \left[A^\dagger A + \frac{1}{2}\right]\hbar\omega$$

If we let it act on its own eigenkets,

$$H|n\rangle = \left[A^\dagger A + \frac{1}{2}\right]\hbar\omega|n\rangle$$

$$= \hbar\omega A^\dagger \sqrt{n}\,|n-1\rangle + \frac{\hbar\omega}{2}|n\rangle$$

$$= \hbar\omega\sqrt{n}\sqrt{n}|n\rangle + \frac{\hbar\omega}{2}|n\rangle$$

$n - 1 + 1$

$$= \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle$$

We see then that the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Position and Momentum

Inverting

$$A = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\omega\hbar}}p$$

$$A^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\,x - \frac{i}{\sqrt{2m\omega\hbar}}p$$

we have

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(A + A^{\dagger}\right)$$

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(A - A^{\dagger}\right)$$

With these, we have a convenient way of evaluating position and momentum expectation values using ladder operators