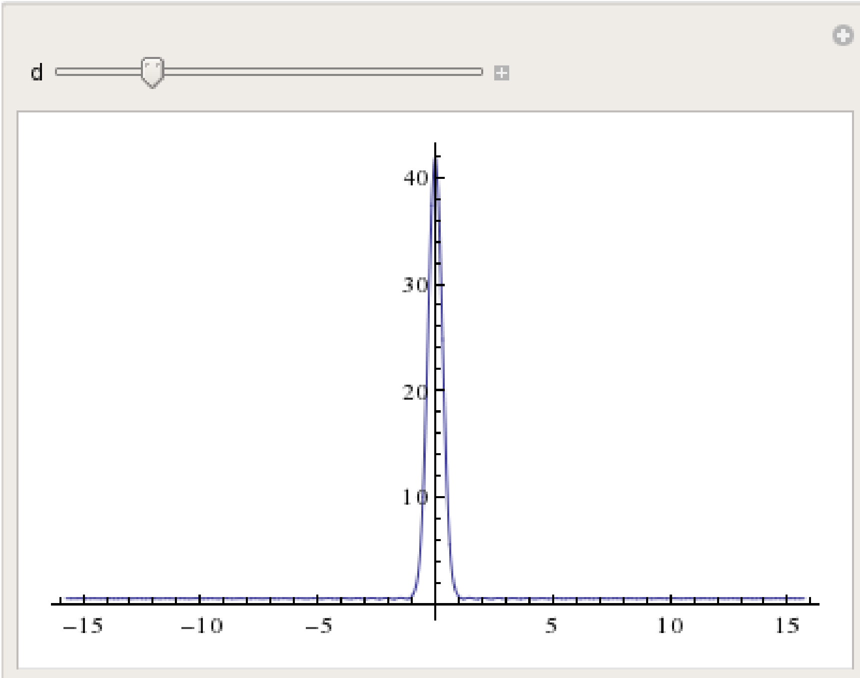
Quantum Mechanics

Robert C. Roleda
Physics Department

Dirac Delta Function

De La Salle University

Definition



The Dirac Delta function (strictly not a function but a distribution) is a function that is zero everywhere except when its argument is zero

$$\delta(x) \begin{cases} = 0, & x \neq 0 \\ \neq 0, & x = 0 \end{cases}$$

$$\delta(x-a) \begin{cases} = 0, & x \neq a \\ \neq 0, & x = a \end{cases}$$

Its value at zero argument is not defined, As such, its definition is complemented by the integral

$$\int_b^c \delta(x-a)dx = \begin{cases} 0, & a \text{ is not between } b \text{ and } c \\ 1, & a \text{ is between } b \text{ and } c \end{cases}$$

Property (i)

Since $\delta(x-a)$ is zero except when $x-a=0$, then

$$\delta(x-a)=\delta(a-x)$$

Property (ii)

Since $\delta(x-a)$ is zero except when $x-a=0$, then for an arbitrary function $f(x)$,

$$\int_b^c f(x)\delta(x-a)dx = \int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)dx$$

where ϵ is an infinitesimal increment, and provided that a is in the region of integration. In this neighborhood of a, $f(x)$ has the fixed value $f(a)$. Hence,

$$\int_b^c f(x)\delta(x-a)dx = \begin{cases} 0, & a \text{ is not between } b \text{ and } c \\ f(a), & a \text{ is between } b \text{ and } c \end{cases}$$

This is the most fundamental, and the most useful property of the Dirac delta-function.

Property (iii)

Using integration by parts,

$$\int f(x)\frac{d\delta(x-a)}{dx}dx = \frac{d}{dx}\int f(x)\delta(x-a)\,dx - \int \frac{df(x)}{dx}\delta(x-a)dx$$

$$= \frac{df(a)}{dx} - \int \frac{df(x)}{dx}\delta(x-a)dx = -\int \frac{df(x)}{dx}\delta(x-a)dx$$

As the derivative of a function is likewise a function, the last expression is

$$-\int \frac{df(x)}{dx}\delta(x-a)dx = -\frac{df(x)}{dx}\bigg|_{x=a} = -f'(a)$$

Thus,

$$\int_b^c f(x)\delta'(x-a)dx = \begin{cases} 0, & a \text{ is not between } b \text{ and } c \\ -f'(a), & a \text{ is between } b \text{ and } c \end{cases}$$

Property (iv)

Suppose we have $\delta(bx)$. First, let $b > 0$. Then by a change of variable $y = bx$,

$$\int f(x)\delta(bx)dx = \int f\left(\frac{y}{b}\right)\delta(y)\frac{dy}{b} = \frac{1}{b}f(0)$$

This has the same effect as

$$\int f(x)\frac{\delta(x)}{b}dx = \frac{1}{b}f(0)$$

We may thus write $\delta(bx) = \delta(x)/b$, for $b > 0$.

On the other hand, for $b < 0$, we may write $\delta(bx) = \delta(-|b|x) = \delta(|b|x)$, where the last relation is due to property (i). Thus, $\delta(bx) = \delta(x)/|b|$ for $b < 0$, and in general,

$$\boxed{\delta(bx) = \frac{\delta(x)}{|b|}}$$

Property (v)

Suppose we have $\delta(g(x))$. If $g(x) = 0$ at $x = a$, then in this neighborhood,

$$g(x) \approx (x-a)\left[\frac{dg}{dx}(a)\right]$$

and by property (iv)

$$\delta\big(g(x)\big) = \frac{\delta(x-a)}{|dg/dx|_{x=a}}$$

If $g(x)$ has several roots $x_1, x_2, \cdots, x_n$,

$$\int f(x)\delta\big(g(x)\big)dx = \sum_{i=1}^{n} \int_{x_i+\epsilon}^{x_i+\epsilon} f(x)\delta\big(g(x)\big)dx$$

since the delta function vanishes except in neighborhoods where its argument is zero. Thus,

$$\boxed{\delta\big(g(x)\big) = \sum_{i-1}^{n} \frac{\delta(x-x_i)}{\left|\frac{dg}{dx}\right|_{x=x_i}}}$$

Three Dimensions

In three-dimensions.

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

For curvilinear coordinates u_1, u_2, u_3,

$$\int \delta^3(\vec{r}) d^3\vec{r} = \int \delta^3(\vec{r})\, h_1 h_2 h_3 du_1 du_2 du_3 = \int \delta(u_1)\, \delta(u_2)\delta(u_3)\; du_1 du_2 du_3$$

requires that

$$\delta^3(\vec{r}) = \frac{\delta(u_1)}{h_1}\frac{\delta(u_2)}{h_2}\frac{\delta(u_3)}{h_3}$$

For example, in spherical coordinates

$$\delta^3(\vec{r}) = \delta(r)\,\frac{\delta(\theta)}{r}\frac{\delta(\varphi)}{r\sin\theta}$$

or

$$\delta^3(\vec{r}) = \delta(r)\frac{\delta(\cos\theta)}{r}\,\frac{\delta(\varphi)}{r}$$