Quantum Mechanics

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Complex Numbers

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Imaginary Numbers

The square of a (real) number is always positive. For example, $2^2 = 4$, $(-2)^2 = 4$.

Square-root is the inverse operation of square. That is, if we take the square root of 4, we should get either +2, or –2.

Likewise, $\sqrt{144} = \pm 12$.

But what about the square root of a negative number? Say $\sqrt{-4}$?

It is definitely not –2, as we have already seen that $(-2)^2 = 4$.

Since the square of all real numbers are positive, the square root of negative numbers form a separate class, which we call *imaginary numbers.*

The number i

A negative number a can be expressed as $-|a|$ or $(-1)|a|$.

Thus, if a is positive, $\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\sqrt{a}$.

For example, $\sqrt{-4} = \sqrt{-1}(2)$.

The "problematic negative sign" can thus be separated out through the factor $\sqrt{-1}$. As a short-hand, we write $\sqrt{-1} = i$.

We see then that

$i^2 = \sqrt{-1}\sqrt{-1} = -1;\qquad i^3 = i^2 i = -i;\qquad i^4 = i^2 i^2 = 1;\qquad i^5 = i^4 i = i$

and in general,

$$i^{n+4} = i^n$$

On the other hand,

$$\frac{1}{i} = \frac{1}{i}\left(\frac{i}{i}\right) = \frac{i}{i^2} = -i$$

Complex Numbers

A complex number is a number with a real and imaginary part. The imaginary part is denoted by the term with an i.

$$Z = a + bi$$

The real part is a,

$$Re(Z) = a$$

and the imaginary part is b

$$Im(Z) = b$$

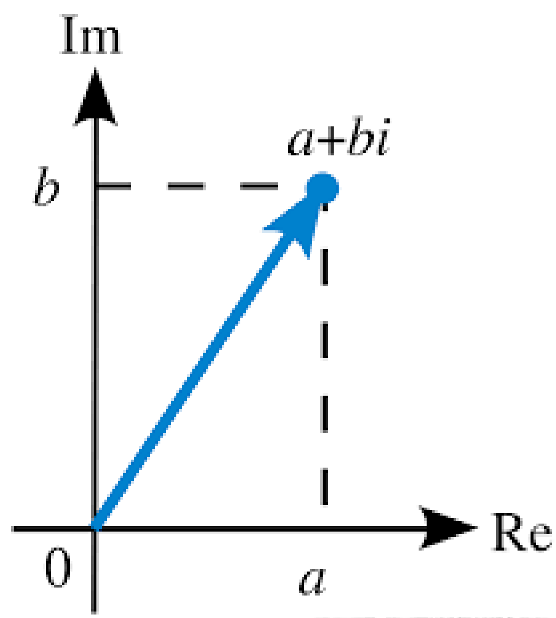
Given two complex numbers

$$W = a + bi$$

$$Z = c + di$$

The sum is

$$W + Z = (a + c) + (b + d)i$$



Products

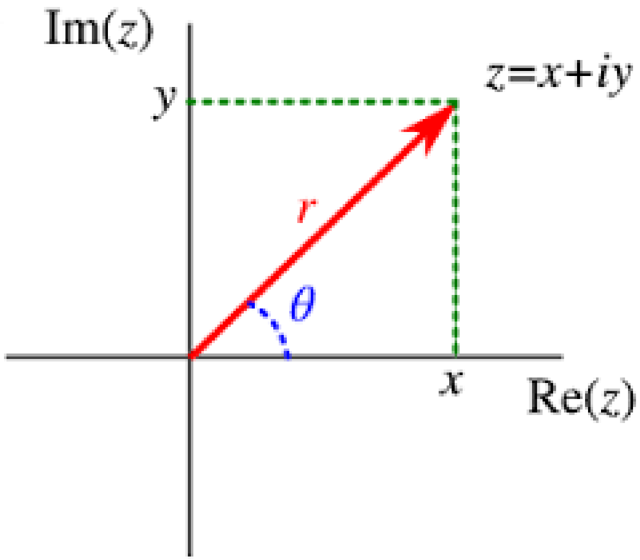
The product of two complex numbers

$$WZ = (a + bi)(c + di)$$
$$= ac + bci + adi + i^2bd$$
$$= (ac - bd) + (bc + ad)i$$

is in general another complex number.

The product is more easily carried out in polar form

$$W = Pe^{i\theta}$$
$$Z = Qe^{i\varphi}$$
$$WZ = PQe^{i(\theta+\varphi)}$$



The complex number z= $x + iy$

is in polar form

$$z = re^{i\theta}$$

where

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

Complex Conjugate

The conjugate of a complex number

$$Z = a + bi$$

is the number where i is replaced by $-i$

$$Z^* = a - bi$$

Thus

$$Z + Z^* = 2a$$

$$Z - Z^* = 2bi$$

and

$$Re(Z) = \frac{Z + Z^*}{2}$$

$$Im(Z) = \frac{Z - Z^*}{2}$$

Norm-Square

Whereas

$$Z^2 = (a + bi)(a + bi) = (a^2 - b^2) + 2abi$$

is a complex number,

$$Z^*Z = (a - bi)(a + bi) = a^2 + b^2$$

is a real number.

In polar form

$$Z^*Z = \left(re^{-i\theta}\right)\left(re^{i\theta}\right) = r^2$$

The product of a complex number with its conjugate is called the *norm-square*

$$|Z|^2 = Z^*Z$$

and this is the length-square of the number in the complex plane.