Quantum Mechanics

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Cramer’s Rule

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Linear Equations

Determinant is a useful tool for solving a set of linear equations.

Let us consider a set of two linear equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

We can solve these equations simultaneously for x by multiplying the first equation by b_2 and subtracting the second equation multiplied by b_2 (thus eliminating y). This gives

$$(a_1b_2 - a_2b_1)x = (c_1b_2 - c_2b_1)$$

which can be written as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Linear Equations

To find y, we do the following

$$a_2(a_1x + b_1y = c_1)$$
$$-a_1(a_2x + b_2y = c_2)$$

to get

$$(a_2b_1 - a_1b_2)y = (c_1a_2 - c_2a_1)$$

which can be written as

$$-(a_1b_2 - a_2b_1)y = -(c_2a_1 - c_1a_2)$$

or

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Cramer's Rule

In general, for a set of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

The solution for a variable x_k is

$$x_k = \frac{\Delta_k}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\Delta_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}$$

*k*th column

This is the **Cramer's Rule**.

Example

Given

$$3x + 2y + z = 11$$
$$2x + 3y + z = 13$$
$$x + y + 4z = 12$$

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 36 + 2 + 2 - 3 - 3 - 16 = 18$$

$$\Delta_1 = \begin{vmatrix} 11 & 2 & 1 \\ 13 & 3 & 1 \\ 12 & 1 & 4 \end{vmatrix} = 132 + 24 + 13 - 36 - 11 - 104 = 18$$

$$\Delta_2 = \begin{vmatrix} 3 & 11 & 1 \\ 2 & 13 & 1 \\ 1 & 12 & 4 \end{vmatrix} = 156 + 11 + 23 - 13 - 36 - 88 = 54$$

$$\Delta_3 = \begin{vmatrix} 3 & 2 & 11 \\ 2 & 3 & 13 \\ 1 & 1 & 12 \end{vmatrix} = 108 + 26 + 22 - 33 - 39 - 48 = 36$$

Cramer's Rule

Thus,

$$x = \frac{\Delta_1}{\Delta} = \frac{18}{18} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{54}{18} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{36}{18} = 2$$