Quantum Mechanics

Robert C. Roleda
Physics Department

Kronecker Delta

De La Salle University

Kronecker Delta

The Kronecker delta is defined by

$$\delta_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}$$

Thus,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$

$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

Leopold Kronecker

Example

Evaluate the expression

$$x_i = \sum_{j=1}^{3} \delta_{ij} y_j$$

where $y_1 = 2, y_2 = -5, y_3 = 3.$

We note that

$$x_i = \delta_{i1} y_1 + \delta_{i2} y_2 + \delta_{i3} y_3$$

For $i = 1,$

$$x_1 = \delta_{11} y_1 + \delta_{12} y_2 + \delta_{13} y_3 = 1 \cdot 2 + 0 \cdot (-5) + 0 \cdot 3 = 2$$

For $i = 2,$

$$x_2 = \delta_{21} y_1 + \delta_{22} y_2 + \delta_{23} y_3 = 0 \cdot 2 + 1 \cdot (-5) + 0 \cdot 3 = -5$$

For $i = 3,$

$$x_3 = \delta_{31} y_1 + \delta_{32} y_2 + \delta_{33} y_3 = 0 \cdot 2 + 0 \cdot (-5) + 1 \cdot 3 = 3$$

Thus, $x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3,$ or generally, $x_i = y_i$

Summing over a Kronecker

The expression

$$x_i = \sum_{j=1}^{3} \delta_{ij} y_j$$

is a sum over all of the values of the index (the dummy index) j. One of the j value will be equal to the free index i. Thus,

$$x_i = \delta_{ii} y_i + \sum_{j \neq i} \delta_{ij} y_j$$

Since $\delta_{ii} = 1$, and $\delta_{ij} = 0$ if $j \neq i$,

$$x_i = y_i$$

Summing over a Kronecker

In general, when summing over a kronecker delta, one only has to replace the summation (dummy) index in the other variables, with the unsummed (free) index of the delta. Thus,

$$\sum_{l} M_{jkl}\delta_{ls} = M_{jks}$$

$$\sum_{l,m} a_{kl}b_{lm}\delta_{ms} = \sum_{l} a_{kl}b_{ls}$$

$$\sum_{j,k} X_{ij}\delta_{jk}Y_{kl} = \sum_{j} X_{ij}Y_{jl} = \sum_{k} X_{ik}Y_{kl}$$

The last two expressions in the last line are exactly the same. That is why the summation index is called a dummy index.

Identities

$$\delta_{ij} = \delta_{ji}$$

$$\sum_{j} \delta_{ij}\delta_{jk} = \delta_{ik}$$

$$\sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$