

Quantum Mechanics

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Kronecker Delta



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Kronecker Delta

The Kronecker delta is defined by

$$\delta_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}$$

Thus,

$$\begin{aligned} \delta_{11} &= \delta_{22} = \delta_{33} = 1 \\ \delta_{12} &= \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0 \end{aligned}$$



Leopold Kronecker



Example

Evaluate the expression

$$x_i = \sum_{j=1}^3 \delta_{ij} y_j$$

where $y_1 = 2, y_2 = -5, y_3 = 3$.

We note that

$$x_i = \delta_{i1}y_1 + \delta_{i2}y_2 + \delta_{i3}y_3$$

For $i = 1$,

$$x_1 = \delta_{11}y_1 + \delta_{12}y_2 + \delta_{13}y_3 = 1 \cdot 2 + 0 \cdot (-5) + 0 \cdot 3 = 2$$

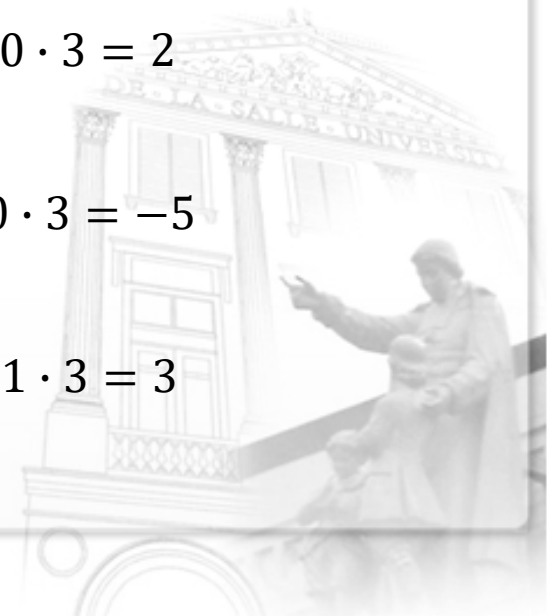
For $i = 2$,

$$x_2 = \delta_{21}y_1 + \delta_{22}y_2 + \delta_{23}y_3 = 0 \cdot 2 + 1 \cdot (-5) + 0 \cdot 3 = -5$$

For $i = 3$,

$$x_3 = \delta_{31}y_1 + \delta_{32}y_2 + \delta_{33}y_3 = 0 \cdot 2 + 0 \cdot (-5) + 1 \cdot 3 = 3$$

Thus, $x_1 = y_1, x_2 = y_2, x_3 = y_3$, or generally, $x_i = y_i$



Summing over a Kronecker

The expression

$$x_i = \sum_{j=1}^3 \delta_{ij} y_j$$

is a sum over all of the values of the index (the dummy index) j . One of the j value will be equal to the free index i . Thus,

$$x_i = \delta_{ii} y_i + \sum_{j \neq i} \delta_{ij} y_j$$

Since $\delta_{ii} = 1$, and $\delta_{ij} = 0$ if $j \neq i$,

$$x_i = y_i$$



Summing over a Kronecker

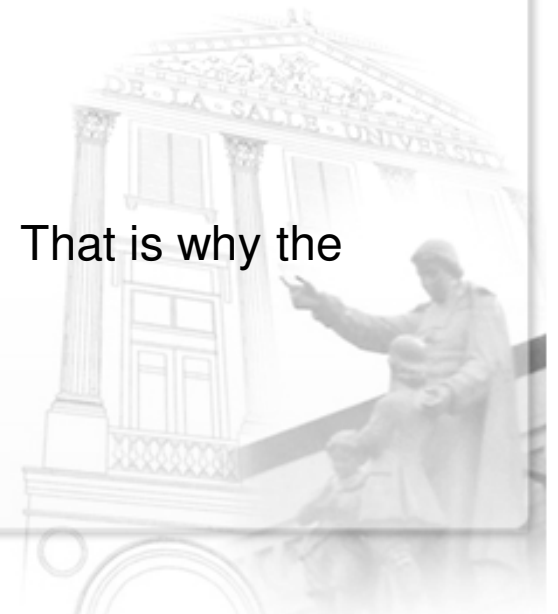
In general, when summing over a Kronecker delta, one only has to replace the summation (**dummy**) index in the other variables, with the unsummed (**free**) index of the delta. Thus,

$$\sum_l M_{jkl} \delta_{ls} = M_{jks}$$

$$\sum_{l,m} a_{kl} b_{lm} \delta_{ms} = \sum_l a_{kl} b_{ls}$$

$$\sum_{j,k} X_{ij} \delta_{jk} Y_{kl} = \sum_j X_{ij} Y_{jl} = \sum_k X_{ik} Y_{kl}$$

The last two expressions in the last line are exactly the same. That is why the summation index is called a dummy index.



Identities

$$\delta_{ij} = \delta_{ji}$$

$$\sum_j \delta_{ij} \delta_{jk} = \delta_{ik}$$

$$\sum_{i=1}^3 \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

