Quantum Mechanics

Robert C. Roleda
Physics Department

Exponentials

De La Salle University

Euler Formula

Euler's formula is a formula that establishes the fundamental relationship between the exponential and the trigonometric functions

$$e^{ix} = \cos x + i \sin x$$

Taking the complex conjugate

$$e^{-ix} = \cos x - i \sin x$$

For example.

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1+i\sqrt{3}}{2}$$

$$e^{-i\pi/4} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1-i}{\sqrt{2}}$$

The Most Beautiful Equation

For $x = \pi$, we get

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

or

$$\boxed{e^{i\pi} + 1 = 0}$$

This is regarded as the most beautiful equation, as it contains the three special numbers $e, i = \sqrt{-1},$ and π, as well as the identify number of addition (zero), and the identity number for multiplication (one).

Trigonometric Functions

If we add

$$e^{ix} = \cos x + i \sin x$$

and

$$e^{-ix} = \cos x - i \sin x$$

we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

If we subtract, we have

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Trigonometric Functions

Thus, the equation

$$y = c_1 e^{ibx} + c_2 e^{-ibx}$$

may be recast as follows

$$y = c_1 e^{ibx} + c_2 e^{-ibx} = c_1(\cos bx + i\sin\ bx) + c_2(\cos bx - i\sin bx)$$
$$= A\cos bx + B\sin bx$$

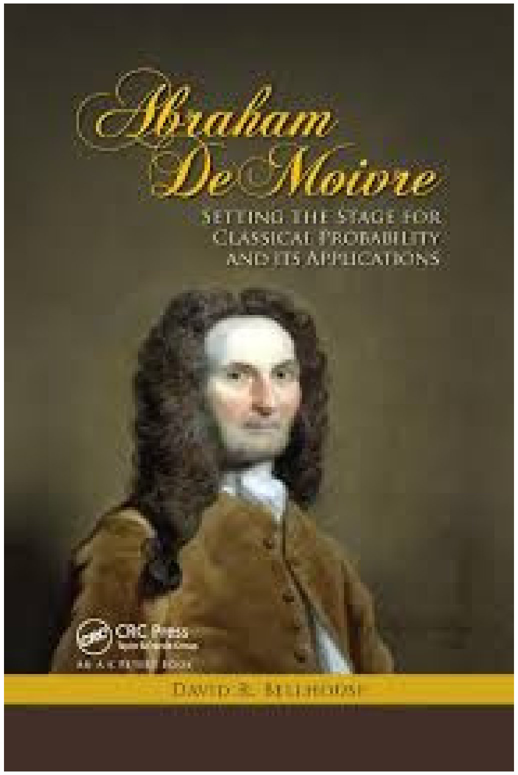
where

$$A = c_1 + c_2; \qquad B = i(c_1 - c_2)$$

are also constant.

It is convenient to recast y this way, especially when we need to evaluate it, or when we apply initial conditions, because values of trigonometric functions are well-known.

De Moivre's Theorem



Euler's formula leads directly to the de Moivre's Theorem.

$$e^{inx} = \cos nx + i \sin nx$$

and

$$e^{inx} = \left(e^{ix}\right)^n = (\cos x + i \sin x)^n$$

Thus,

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

Multiple Angle Formulas

De Moivre's formula is convenient for deriving multiple angle formulas. For example, for $n = 2$,

$$(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x$$

This leads to

$$(\cos x)^2 - (\sin x)^2 + 2i \sin x \cos x = \cos 2x + i \sin 2x$$

Comparing real and imaginary parts gives us

$$\cos 2x = (\cos x)^2 - (\sin x)^2$$

$$\sin 2x = 2 \sin x \cos x$$

Power Formulas

De Moivre's formula is also useful for deriving power forumulae. For example, for $n = 3$,

$$(\cos x + i\sin x)^3 = (\cos x)^3 + 3i(\cos x)^2 \sin x - 3\cos x\,(\sin x)^2 - i(\sin x)^3 =$$
$$= \cos 3x + i \sin 3x$$

$$(\cos x - i\sin x)^3 = (\cos x)^3 - 3i(\cos x)^2 \sin x - 3\cos x\,(\sin x)^2 + i(\sin x)^3 =$$
$$= \cos 3x - i \sin 3x$$

Adding gives us

$$(\cos x)^3 - 3\cos x\,(\sin x)^2 = \cos 3x$$

Which can be recast as

$$4(\cos x)^3 = \cos 3x + 3\cos x$$

or

$$(\cos x)^3 = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

Hyperbolic Functions

If the argument of the exponential is real, we have the hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

If we subtract, we have

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Thus,

$$e^{\pm x} = \cosh x \pm \sinh x$$

and

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x}$$

yields

$$(\cosh x)^2 - (\sinh x)^2 = 1$$