Quantum Mechanics

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Classical Probability

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Probability

Probability is a measure of the likelihood that an event would occur.

If an event is certain to occur, its probability is 1. If an event is impossible, then its probability is 0.

Probability of an event is a number between 0 and 1.

$$0 \leq p(x) \leq 1$$

When you toss a coin, there are two possible outcomes – head or tail. If the coin is fair, then a head would have the same likelihood of appearing as a tail. If *p* is the probability that a Head appears, and *q* is the likelihood that a Tail appears, then $p = q = \frac{1}{2}$.

Note that $p + q = 1$. This is because when you toss a coin, it is certain that either a Head or a Tail would occur. Certainty is expressed by a probability of 1, while the probability that Head OR Tail would occur is expressed by the addition.

Complement

The opposite or complement of an event *A* is the event NOT A. That is, the event of *A* not occurring. The complement of *A* may be denoted by A^c.

For a coin toss, the complement of Head is a Tail. The probability of a Tail or the complement of a Head is

$$P(H^c) = P(T) = q = 1 - p = 1 - P(H)$$

When you roll a six-sided die, the probability that a 3 would appear is

$$P(3) = \frac{1}{6}$$

The probability that what would appear is not a 3 is

$$P(not\ 3) = 1 - P(3) = \frac{5}{6}$$

Independent Events

Two events are independent if the occurrence of one does not affect that occurrence of the other.

If two coins are flipped, we may measure the joint probability of events *A* AND *B* occurring. If the two coins are distinguishable, for example, the first coin is 25 centavo and the second is a peso coin, the different possibilities and the corresponding probabilities may be enumerated

Coin 1	H	H	T	T
Coin 2	H	T	H	T
$P(A \cap B)$	$P(H \cap H) = 1/4$	$P(H \cap T) = 1/4$	$P(T \cap H) = 1/4$	$P(T \cap T) = 1/4$

We note that for each coin $P(H) = 1/2$, and $P(T) = 1/2$, and that for example

$$P(H \cap T) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4}$$

Independent Events

If we roll two 3-sided dice that are distinguishable, say one is red the other yellow, the different possibilities are

Die 1	1	1	1	2	2	2	3	3	3
Die 2	1	2	3	1	2	3	1	2	3

The joint probability that the first die cast 3 and the second not 3 is

$$P(3 \cap 3^c) = 2/9$$

as this corresponds to the events (3,1) and (3,2) out of the 9 possibilities.

We note that for each die, $P(3) = 1/3$, and $P(not\ 3) = 2/3$, and that

$$P(3 \cap 3^c) = P(3)P(3^c) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2/9$$

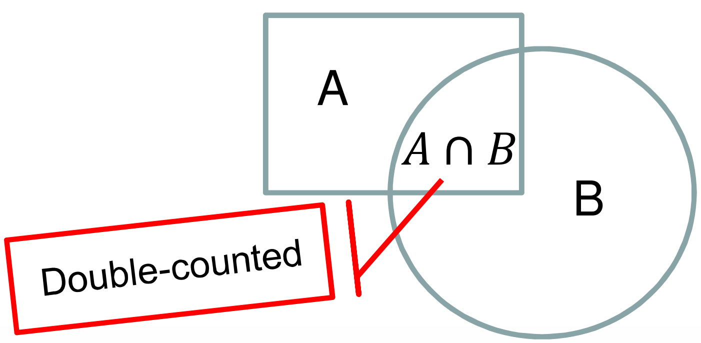
Since the two dice presumably do not affect what would occur on the other,

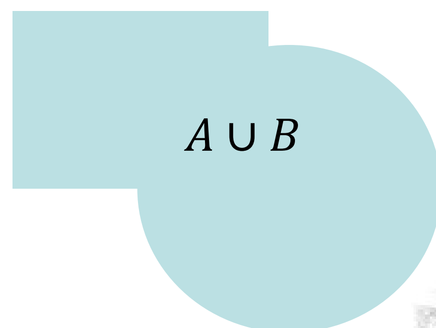
$$P(A \cap B) = P(A)P(B)$$

Union and Intersection

The probability of either *A* OR *B* occurring is

$$P(A \text{ } or \text{ } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$





Mutually Exclusive Events

If in a single throw (trial), either *A* or *B* but never both occur, the two events are said to be mutually exclusive. In this case,

$$P(A\ and\ B) = P(A \cap B) = 0$$

Then the probability of either *A* OR *B* occurring is

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$$

For example, the likelihood of your rolling a 2 or an odd number with a six-sided die is

$$P(2\ or\ odd) = P(2) + P(odd) = \frac{1}{6} + \frac{1}{2} = \frac{4}{6}$$

as 2 or odd is the set (1,2,3,5) or 4 out of the 6 possible outcomes.

Non-Mutually Exclusive Events

For non-mutually exclusive events,

$$P(A\ or\ B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

For example, if you draw a card from a regular deck of 52 cards, the probability of getting a spade or a face card (J,K,Q) or one that is both

$$P(spade\ or\ face) = P(spade) + P(face) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$$

since there are 13 spades, 12 face cards, and 3 face cards that are spades.

Total Probability

If A_i for $i = 1,2,\ldots,n$ is the set of all events that can happen, then

$$\sum_{i=1}^{n} P(A_i) = 1$$

Thus, for a six-sided die, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$, and

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

This is simply a statement that if we roll a die, we are sure that either a 1 or a 2 or a 3 or a 4 or a 5 or a 6 will appear.

Note that the complement of 3 is 1 or 2 or 4 or 5 or 6. Thus,

$$P(3^c) = P(1) + P(2) + P(4) + P(5) + P(6) = 5/6$$

as discussed earlier.

Empirical Probability

We have so far assumed that in the coin toss, the probability of a Head is equal to the probability of a Tail and that both are equal to ½. Likewise, the probability of each face of a six-sided die is 1/6. These probabilities are based on a presumption of fairness, and determined theoretically.

Probability may also be determined by carrying out numerous trials of an experiment. The probability of an event A can be inferred from the number of times n that such event occurs in N tries

$$P(A) = \frac{n}{N}$$

This is the *frequentist probability.*

Mean

Suppose we do an experiment where the possible values are whole numbers from 1 to 4. And suppose that after ten trials, the values obtained are 1,2,4,2,3,1,2,4,2,1

To report the result of the experiment, we take a representative value which often time is the *mean* or *average*. The arithmetic mean in this case is

$$m = \frac{1+2+4+2+3+1+2+4+2+1}{10} = 2.2$$

If we denote the value from each trial as x_i, and the number of trials as N, the arithmetic mean $\bar{x}$ is in general

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$$

Mean

The mean may also be obtained by grouping each value together,

$$m = \frac{1 + 1 + 1 + 2 + 2 + 2 + 2 + 3 + 4 + 4}{10} = 2.2$$

or multiplying each value by the number of times it occurred

$$m = \frac{1f(1) + 2f(2) + 3f(3) + 4f(4)}{10} = \frac{1(3) + 2(4) + 3(1) + 4(2)}{10} = 2.2$$

If the experiment is repeated many times, the probability for each outcome may be evaluated from its frequency of occurrence. Thus,

$$P(1) = \frac{f(1)}{N} = \frac{3}{10}, P(2) = \frac{f(2)}{N} = \frac{4}{10}, P(3) = \frac{f(3)}{N} = \frac{1}{10}, P(4) = \frac{f(4)}{N} = \frac{2}{10}$$

The mean may then be calculated using

$$\bar{x} = \sum_{i=1}^{N} x_i P(x_i)$$

Expectation Values

The mean is the representative value of a set of measurements. Suppose at the outset, the probability of each value is known. Thus, even before an experiment is conducted, one would be able to predict what the mean would be using

$$\bar{x} = \sum_{i=1}^{N} x_i P(x_i)$$

As the mean value in this case is not determined from actual measurements, but is predicted using probability theory, it is called the *expectation value*.

If the values are continuous, we may define a probability density $p(x)$ such that the probability over a range of values $a < x < b$ is

$$P(a < x < b) = \int_a^b p(x)dx$$

Expectation Values

The expectation value of x is then defined as

$$\langle x \rangle = \int x p(x) dx$$

where the integration is over the entire range of values of x .

One may also determine expectation values of functions $y(x)$ via

$$\langle y(x) \rangle = \int y(x) p(x) dx$$

For example,

$$\langle x^2 \rangle = \int x^2 p(x) dx$$

In general,

$$\langle x^m \rangle = \int x^m p(x) dx$$

are called the m-th moment of x .

Variance

Variance is a measure of the spread of values and it is defined as

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N}$$

Expanding the binomials

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{N}$$

As the mean value $\bar{x}$ is a fixed value,

$$\sigma^2 = \frac{\sum_{i=1}^{N} x_i^2}{N} - 2\bar{x}\frac{\sum_{i=1}^{N} x_i}{N} + \bar{x}^2\frac{\sum_{i=1}^{N} 1}{N}$$

This yields

$$\sigma^2 = \langle x^2 \rangle - 2\langle x \rangle\langle x \rangle + \langle x \rangle^2$$

or

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

While the second moment relates to the variance, the third moment is related to skewness, and the fourth moment to the kurtosis.